STOR 435 Midterm 1 Study Guide

Counting · Events · Probability rules · Conditional probability

0. Before you start

WHEN. Tuesday, September 22, in class. 50 minutes. Check Canvas in case the date was moved.

FORMAT. 12 to 15 multiple-choice questions, one point each. Closed book. Bring a calculator that does factorials and e^x. Answers are usually a number you must place in an interval, or a formula you must recognize.

TOPICS. Four ideas, in this order: how to count outcomes, how to describe events with sets, the rules probabilities obey, and how probabilities change when you learn something.

How to use this guide

- **Read sections 1 to 5 in order.** Each idea builds on the one before. Blue boxes are the formulas you must know. Orange boxes are the mistakes that cost points. Green boxes tell you how a question will probably be phrased.
- **Do every "Check yourself" question with pen and calculator** before opening the answer. On the exam you need the actual number, not just the setup.
- **Section 6** is material taught just before the exam that is officially for Midterm 2. Read it if you have time.
- **Section 8** is a full practice exam with worked answers. **Section 9** is a one-page summary for the night before.

If you have one day

1. Sections 1 and 2, about 45 minutes. Do the counting questions.
2. Sections 3 and 4, about 45 minutes. Be able to write the three-event inclusion–exclusion formula from memory.
3. Section 5, about 30 minutes. Do the multiplication-rule questions.
4. Take the practice exam in section 8 in 50 minutes, then read every solution.
5. Read the summary in section 9 before bed and again before the exam.

1. Three math facts you need

These are not probability, but they appear on the exam as quick questions and inside harder ones.

1.1 Geometric series

ADDING POWERS OF A NUMBER. For $a \neq 1$,

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

If $-1 < a < 1$ the terms shrink, and the infinite sum is

$$1 + a + a^2 + \cdots = \frac{1}{1 - a}, \qquad a + a^2 + a^3 + \cdots = \frac{a}{1 - a}$$

How to use it. Factor out whatever every term has in common until what is left looks like $1 + a + a^2 + \cdots$, then apply the formula.

- $1 + (1/4)^2 + (1/4)^4 + (1/4)^6 + \cdots$ has ratio $a = 1/16$, so it equals $1/(1 - 1/16) = 16/15$.
- $(1/3)^2(1/4)^2 + (1/3)^3(1/4)^4 + (1/3)^4(1/4)^6 + \cdots$: the n-th term is $(1/3)(1/3)^n(1/16)^n = (1/3)(1/48)^n$. Summing from $n = 1$ gives $(1/3) \cdot (1/48)/(1 - 1/48) = 1/141 \approx 0.0071$.

1.2 The number e

TWO WAYS E SHOWS UP.

$$\Sigma_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x \qquad \lim_{n\to\infty} \left(1 - \frac{a}{n}\right)^n = e^{-a}$$

- $\Sigma\ (-2)^n/n! = e^{-2} \approx 0.1353$. $\Sigma\ (-3)^n/n! = e^{-3} \approx 0.0498$.
- $\lim\ (1 - 3/n)^n = e^{-3} \approx 0.0498$. $\lim\ (1 + 2/n)^n = e^2$.

WATCH OUT. $\lim\ (1 - 0.00001)^n = 0$, not $e^{-\text{something}}$. In the e limit the base moves toward 1 as n grows. A fixed base below 1 raised to a growing power goes to zero.

1.3 Factorials and the calculator

- $n! = n \times (n - 1) \times \cdots \times 2 \times 1$, and $0! = 1$.
- A partial product like $10 \times 9 \times 8$ equals $10!/7!$. In general $n(n - 1) \cdots (n - r + 1) = n!/(n - r)!$.
- Calculators overflow around 70!. Never compute 365!. Multiply fractions one at a time instead: $(364/365)(363/365) \cdots$.

- Cancel before multiplying: $\binom{300}{3} = (300 \cdot 299 \cdot 298)/6$.
- Find the nCr key on your calculator and practice with it before the exam.

1.4 Check yourself

1. $\Sigma_{n=0}^{\infty} (-1)^n/n!$ lies in which interval: (0.3, 0.4], (0.4, 0.5], or (0.5, 0.6]?

$e^{-1} \approx 0.3679$, so (0.3, 0.4].

2. Evaluate $1 + 1/3 + 1/9 + 1/27 + \cdots$.

Ratio 1/3: $1/(1 - 1/3) = 1.5$.

3. Evaluate $\lim_{n\to\infty} (1 - 2/n)^n$ to 4 decimals.

$e^{-2} \approx 0.1353$.

4. Evaluate $\Sigma_{n=1}^{\infty} (1/2)^n(1/5)^n$.

Each term is $(1/10)^n$. From n = 1: $(1/10)/(1 - 1/10) = 1/9 \approx 0.1111$.

5. Write $12 \times 11 \times 10 \times 9$ with factorials, then compute $\binom{12}{4}$.

$12!/8! = 11{,}880$. Divide by 4! to get $\binom{12}{4} = 495$.

2. Counting

Why counting comes first. When every outcome is equally likely, the probability of an event is simply (outcomes in the event) ÷ (all outcomes). To use that, you must be able to count outcomes without listing them. This section gives you five counting tools and the patterns for combining them.

2.1 Multiply the stages

MULTIPLICATION PRINCIPLE. If a process happens in stages, and stage 1 has n_1 options, stage 2 has n_2 options (no matter what happened in stage 1), and so on, the total number of outcomes is $n_1 \cdot n_2 \cdot \cdots n_r$.

The only requirement is that the *number* of options at each stage does not depend on earlier choices. The options themselves may change (after using the letter A you cannot use it again), as long as the count stays the same.

- Two-letter strings, repeats allowed: $26 \cdot 26 = 676$. No repeats: $26 \cdot 25 = 650$.
- License plate with 3 letters then 4 digits: $26^3 \cdot 10^4 = 175{,}760{,}000$. With no repeated letter or digit: $(26 \cdot 25 \cdot 24)(10 \cdot 9 \cdot 8 \cdot 7) = 78{,}624{,}000$.
- A pizza has 4 sizes and any subset of 8 toppings: $4 \cdot 2^8 = 1024$ options. Two pizzas ordered one after the other: 1024^2.

NUMBER OF SUBSETS. A set of n objects has 2^n subsets, including the empty set. Each object is either in or out: n stages, 2 options each.

Example: a forbidden pair. How many subsets of a 10-student class do not contain both Brad and John? All subsets: $2^{10} = 1024$. Subsets containing both: the other 8 students are free, $2^8 = 256$. Answer $1024 - 256 = 768$.

COUNT EVERYTHING, SUBTRACT THE FORBIDDEN. This is the most reused trick in the whole section. Whenever a condition says "not both", "at least one", or "no two the same", count the opposite and subtract.

2.2 Permutations: order matters

ARRANGING N DISTINCT OBJECTS IN A ROW. $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ ways. Arranging only r of them: $n(n-1) \cdots (n-r+1) = n!/(n-r)!$.

Example: books in blocks. 12 distinct books: 5 math, 3 sci-fi, 2 cookbooks, 1 environment, 1 teaching. Books of the same type must sit together. Treat each type as one block: 5 blocks in 5! orders. Inside the blocks: 5! · 3! · 2! · 1! · 1!. Total 5! · 5! · 3! · 2! = 172,800.

"Must be together". Glue the items into one block, arrange the blocks, then arrange inside each block.

Example: visiting cities. Visit 4 of the 6 largest US cities. If the order of visits matters: $6 \cdot 5 \cdot 4 \cdot 3 = 360$. If it does not: $\binom{6}{4} = 15$. Same story, two different questions.

2.3 Combinations: order does not matter

Choosing a group of r from n.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Read "n choose r". $\binom{n}{0} = 1$ and $\binom{n}{r} = \binom{n}{n-r}$.

Where it comes from. Ordered lists of r objects: $n!/(n-r)!$. Every group of r objects shows up r! times in that list, once per ordering. Divide by r! to count each group once.

Example: committee with a feud. From 5 juniors and 7 seniors, choose 3 juniors and 2 seniors. Two seniors refuse to serve together.

1. Ignore the feud: $\binom{5}{3}\binom{7}{2} = 10 \cdot 21 = 210$.
2. Count committees with both feuding seniors: their two seats are taken, choose the 3 juniors: $\binom{5}{3} = 10$.
3. Subtract: $210 - 10 = 200$.

Example: party guests. Invite 5 of 8 friends; two of them will not both attend. $\binom{8}{5} - \binom{6}{3} = 56 - 20 = 36$ possible groups.

Example: "at least" with cases. Choose 6 of 20 courses: 5 Statistics, 6 Physics, 9 other. Need at least 2 Statistics and at least 3 Physics. List the possible (Stats, Physics, Other) splits: (2,3,1) gives $\binom{5}{2}\binom{6}{3}\binom{9}{1} = 1800$; (2,4,0) gives $\binom{5}{2}\binom{6}{4} = 150$; (3,3,0) gives $\binom{5}{3}\binom{6}{3} = 200$. Total 2150.

Example: no two defectives together. 100 batteries in a line, 80 working and 20 defective, same-type batteries identical. How many orderings have no two defectives adjacent? Lay out the 80 working ones; that creates 81 gaps (before, between, after). Put at most one defective in a gap: choose 20 of 81 gaps, $\binom{81}{20}$.

"No two adjacent". Place the other items first, then choose gaps.

2.4 The binomial theorem

BINOMIAL THEOREM.

$$(x + y)^n = \Sigma_{r=0}{}^{n} \binom{n}{r} x^r y^{n-r}$$

$(x + y)^n$ is n copies of $(x + y)$ multiplied together. To produce $x^r y^{n-r}$, pick x from exactly r of the copies: $\binom{n}{r}$ ways.

Setting $x = y = 1$ gives $\Sigma_r \binom{n}{r} = 2^n$, the subset count again.

2.5 Multinomial coefficients: several groups, or repeated letters

MULTINOMIAL COEFFICIENT. For $n_1 + n_2 + \cdots + n_r = n$,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

This one number answers two different questions:

Splitting into labeled groups.
The number of ways to divide n distinct objects into groups of sizes $n_1, \ldots, n_r$ when the groups have names or jobs. It equals $\binom{n}{n_1}\binom{n-n_1}{n_2} \cdots$ (fill group 1, then group 2 from what is left, and so on).

Arranging objects with repeats.
The number of ways to arrange n objects in a row when n_1 are identical, n_2 are identical, and so on. Start with n! and divide by $n_i!$ for each type, because swapping identical objects changes nothing.

- **MISSISSIPPI:** 11 letters, I×4, S×4, P×2, M×1: 11!/(4! 4! 2!) = 34,650. **SEATTLE:** 7!/(2! 2!) = 1260.
- **Bridge:** 52 cards into four players' hands of 13: $52!/(13!)^4$.
- **Chess:** 7 games, each a win (1 point), draw (½) or loss (0) for player A; A finishes with 4 points. The number of draws must be even. 0 draws: 4W 3L, 7!/(4! 3!) = 35. 2 draws: 3W 2D 2L, 7!/(3! 2! 2!) = 210. 4 draws: 2W 4D 1L, 105. 6 draws: 1W 6D, 7. Total 357.

Labeled or unlabeled groups?

Labeled. 8 firefighters into 4 pairs: truck, phones, cleaning, standby. The jobs name the groups: $8!/(2!)^4 = 2520$.

Unlabeled. 8 firefighters into 4 pairs, no jobs. The same four pairs can be labeled "pair 1–4" in 4! ways, all the same split. Divide: 2520/4! = 105.

WATCH OUT. Splitting 90 soldiers into three groups of 30 for three *different* jobs is $\binom{90}{30,\,30,\,30}$, with no 3! anywhere. Divide by (number of groups)! only when the groups are interchangeable, meaning the problem never distinguishes group 1 from group 2.

2.6 Seating problems

8 animals sit in 8 chairs, 4 on each side of a table. Every chair is different, so a seating is an arrangement of 8 animals: $8! = 40{,}320$ in total.

- **6 cats and 2 dogs, dogs not on the same side.** Seat dog 1 anywhere (8), dog 2 on the opposite side (4), the cats fill the rest (6!): $8 \cdot 4 \cdot 720 = 23{,}040$. Check by subtraction: same side = 2 sides $\times\ 4 \cdot 3$ chairs for the dogs $\times\ 6! = 17{,}280$, and $40{,}320 - 17{,}280 = 23{,}040$.
- **4 dogs and 4 cats, probability all dogs on one side.** Favorable: choose the dogs' side (2), arrange the dogs there (4!), arrange the cats opposite (4!). Probability $2(4!)^2/8! \approx 0.0286$.

2.7 Which formula?

Situation	Count	Test
Several choices in a row	$n_1 \cdot n_2 \cdots n_r$	Fixed number of options at each stage?
Arrange all n distinct things	$n!$	Does swapping two give a new outcome?
Arrange r of n distinct things	$n!/(n-r)!$	Order matters, not everything used.
Pick a group of r from n	$\binom{n}{r}$	Order does not matter.
Split into labeled groups	$n!/(n_1! \cdots n_r!)$	Groups have names? If not, also divide by (groups)!.
Arrange with repeated items	$n!/(n_1! \cdots n_r!)$	Some items identical?
Any subset	2^n	Each item in or out.
"These two not both"	all − both	Count the complement.
"No two adjacent"	choose gaps	Place the others first.
"Must be together"	glue a block	Arrange inside the block too.

WATCH OUT. Never mix an ordered count with an unordered one in the same fraction. If the numerator counts ordered sequences, the denominator must too. Both ways give the same probability, but only if you are consistent.

2.8 Check yourself

1. How many 5-letter strings from 26 letters have no repeated letter?

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7{,}893{,}600.$

2. 5 tall players stand in a row behind 7 short players seated in a row. How many photos?

$5! \cdot 7! = 604{,}800.$

3. How many arrangements of the letters in BANANA?

A×3, N×2, B×1: $6!/(3!\,2!) = 60.$

4. Box 1 has 4 red and 6 blue balls; box 2 has 16 red and some blue. All balls are distinct. Drawing one from each box, there are 124 ways to get two of the same color. How many blue balls are in box 2?

$4 \cdot 16 + 6x = 124$, so $x = 10.$

5. 12 students form three project teams of 4 with different topics. How many ways? What if the teams are interchangeable?

Labeled: $12!/(4!)^3 = 34{,}650$. Unlabeled: divide by 3!, giving 5775.

6. 5 people sit in a row; Ann and Ben must be next to each other. How many seatings?

Glue Ann and Ben: 4 units in 4! ways, times 2 for their order: 48.

7. A committee of 4 from 6 men and 5 women must have at least 3 women. How many?

3 women and 1 man: $\binom{5}{3}\binom{6}{1} = 60$. 4 women: 5. Total 65.

8. What is the coefficient of x^3y^5 in $(x + y)^8$?

$\binom{8}{3} = 56.$

3. Sample spaces and events

The idea. Before you can find the probability of something, you must say exactly what that something is. The sample space is the list of everything that could happen. An event is a part of that list. "Or", "and", "not" become union, intersection, complement. Exam questions test whether you can move between words, set notation, and pictures without slipping.

3.1 Sample spaces

SAMPLE SPACE S. The set of all possible outcomes of an experiment.

The sample space depends on what you write down. Flip a coin twice. Recording each flip gives S = {HH, HT, TH, TT}. Recording only the number of heads gives S = {0, 1, 2}. Same experiment, different sample space. Read carefully what is being recorded.

- Six runners finish with no ties: S is the set of all orderings, and its size is 6! = 720. The sample space is the set; 720 is its size.
- Flip a coin until the first head: S = {H, TH, TTH, TTTH, …}. Infinite, but listable.
- Lifetime of a light bulb in hours: S = $[0, \infty)$. Continuous.
- Draw 2 marbles from {2 red, 1 green, 1 blue}, recording colors in order. With replacement there are 9 outcomes (RR, RG, RB, GR, GG, GB, BR, BG, BB). Without replacement, GG and BB are impossible, leaving 7. RR stays possible because there are two reds.
- Roll a black die and note even/odd, then a white die and note even/odd: S = {(E,E), (E,O), (O,E), (O,O)}. Ordered pairs, black die first. Not the faces, not unordered pairs.
- Pick one of 27 flags on each of 3 days, repeats allowed: S = all triples of flags, $27^3 = 19{,}683$ outcomes.

3.2 Events and the three operations

EVENT. Any subset of S. "E occurs" means the outcome is in E.

Union E ∪ F.
Outcomes in E, in F, or in both. "E or F occurs." The "or" includes both.

Intersection E ∩ F.
Outcomes in both. "E and F occur."

Complement E^c.
Outcomes not in E. "E does not occur." $S^c = \varnothing$.

Empty event ∅.
Contains no outcomes; cannot occur. If E ∩ F = ∅, E and F are **disjoint** (mutually exclusive): they cannot both happen.

Containment E ⊆ F.

Every outcome of E is in F. If E happens, F happens. E = F means each contains the other.

For several events, $\cup_j E_j$ means "at least one occurs" and $\cap_j E_j$ means "all occur".

Example. S = {HH, HT, TH, TT}. E = first coin is H = {HH, HT}. F = the coins differ = {HT, TH}. Then E ∪ F = {HH, HT, TH}, E ∩ F = {HT}, E^c = {TH, TT}.

> AN EVENT WITH EXTRA CONDITIONS IS NEVER MORE LIKELY. If B ⊂ A then P(B) ≤ P(A). "Linda is a bank teller and a feminist" cannot be more probable than "Linda is a bank teller", however well it fits the story. The die sequence GRGRRR ends with RGRRR, so whenever GRGRRR appears, RGRRR has appeared too; the longer sequence cannot be more likely.

3.3 Rules for combining events

Rule	Statement	In words
Commutative	E ∪ F = F ∪ E, E ∩ F = F ∩ E	Order does not matter.
Associative	(E ∪ F) ∪ G = E ∪ (F ∪ G), same for ∩	Grouping does not matter when the operation is the same.
Distributive	(E ∪ F) ∩ G = (E ∩ G) ∪ (F ∩ G) (E ∩ F) ∪ G = (E ∪ G) ∩ (F ∪ G)	Like a(b + c) = ab + ac, but works both ways.
De Morgan	$(E \cup F)^c = E^c \cap F^c$ $(E \cap F)^c = E^c \cup F^c$	"Not (E or F)" = neither. "Not (E and F)" = at least one fails. Complement swaps ∪ and ∩.

De Morgan works for any number of events: the complement of "at least one E_j" is "no E_j", and the complement of "all E_j" is "at least one fails".

A useful split. $E \cup F = E \cup (E^c \cap F)$. "E or F" is "E" plus "F but not E", and those two pieces do not overlap. This is how the addition rule in section 4 is proved.

3.4 Turning words into events

For three events A, B, C:

Words	Event
A and B but not C	$A \cap B \cap C^c$
Only A	$A \cap B^c \cap C^c$
Exactly one	$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

Exactly two	$(A \cap B \cap C^c) \cup (A \cap C \cap B^c) \cup (B \cap C \cap A^c)$
At least two	$(A \cap B) \cup (A \cap C) \cup (B \cap C)$
At most one	$(A \cap B)^c \cap (A \cap C)^c \cap (B \cap C)^c$ (no pair occurs together)
At least one	$A \cup B \cup C$
None	$A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$

Method: list the cases that satisfy the words, write each case as an intersection (what happens ∩ what does not), then join the cases with ∪.

3.5 Simplifying a messy expression

The exam gives a complicated expression and asks which simple event it equals. Two approaches.

Reason in words. $(A \cup B) \cap (A \cup C) \cap (B^c \cap C^c)$. The last part says B and C both fail. Then "A or B" forces A, and "A or C" forces A. The whole thing is $A \cap B^c \cap C^c$: only A.

Use the rules. $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c)$. First, $(A \cup B) \cap (A^c \cup B) = B$: if B occurs both parts hold; if B fails you would need A and A^c at once. Then $B \cap (A \cup B^c) = (B \cap A) \cup (B \cap B^c) = A \cap B$.

> When stuck, test outcomes. Make a table with one row for each yes/no combination of A, B, C (eight rows) and mark which rows satisfy the expression. It takes two minutes and is safer than pushing symbols.

3.6 Worked examples

Colored balls. Draw 3 balls without replacement from 5 black and 2 red, recording colors in order. RRR is impossible, so S = {BBB, BBR, BRB, RBB, BRR, RBR, RRB}. A_2 = second ball black = {BBB, BBR, RBB, RBR}. $A_3{}^c$ = third ball red = {BBR, BRR, RBR}. $A_2 \cap A_3{}^c$ = {BBR, RBR}.

Integers from −7 to 7. A = {N ≤ 3}, B = {2 ≤ N ≤ 10} = {2,…,7} within range, C = {N even and positive} = {2, 4, 6}.

- $A \cap B \cap C = \{2\}$.
- $A^c \cup (B \cap C^c) = \{4,5,6,7\} \cup \{3,5,7\} = \{3,4,5,6,7\}$.
- $(A \cup B) \cap C^c$: A ∪ B is everything, so remove {2,4,6}: {−7,…,1,3,5,7}.
- $(A \cap B) \cap (A \cup C)^c$: $A \cap B = \{2,3\}$; $(A \cup C)^c = \{5,7\}$; no overlap, ∅.

Coin until the first tail. S = {T, HT, HHT, …}. E_k = at least k tosses needed. $E_4 \cap E_8{}^c$ = at least 4 but fewer than 8 tosses = {HHHT, HHHHT, HHHHHT, HHHHHHT}.

Elevator. Three people each pick a floor from 1 to 10: S has $10^3 = 1000$ outcomes. E = they pick three consecutive floors. The floor set is {k, k+1, k+2} for k = 1 to 8, and the three people can be matched to those floors in 3! ways: $|E| = 8 \cdot 6 = 48$.

3.7 Check yourself

1. Roll two dice, recording the ordered pair. How many outcomes in S? How many in E = {sum is 7}?

$|S| = 36$. $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, $|E| = 6$.

2. Write "at least one of A, B fails" two ways.

$A^c \cup B^c = (A \cap B)^c$.

3. Simplify $(E \cup F) \cap (E \cup F^c)$.

$E \cup (F \cap F^c) = E \cup \varnothing = E$.

4. S is all 26^2 two-letter strings. E = letters differ, F = one letter is A. Describe $E \cap F$.

Strings with exactly one A and a different second letter: AB, AC, …, AZ, BA, CA, …, ZA. Not AA.

5. Is "disjoint" the same as "complements"?

No. Disjoint: $E \cap F = \varnothing$. Complements: also $E \cup F = S$. {HH} and {TT} are disjoint but not complements.

6. Which is $(A^c \cap B^c)^c$: "neither A nor B", "A and B", or "at least one of A, B"?

At least one. By De Morgan it equals $A \cup B$.

7. Which of these equals $E \cup F$: $E \cup (E^c \cup F)$, $E \cup (E^c \cap F)$, $E \cap (E^c \cap F)$?

$E \cup (E^c \cap F)$. The first equals S; the third is $\varnothing$.

4. The rules of probability

The idea. Probability is not defined as "long-run frequency" (how long is long? and you cannot repeat "Apple stock rises next year"). Instead, three rules say what any probability assignment must satisfy, and everything else follows. You need to: check whether an assignment obeys the rules, use the consequences (complement, inclusion–exclusion), and compute probabilities by counting when outcomes are equally likely.

4.1 The three axioms

AXIOMS.

- **1.** $0 \le P(E) \le 1$ for every event E.
- **2.** $P(S) = 1$.
- **3.** If E_1, E_2, … are disjoint, then $P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$.

Axiom 3 in words: if events cannot overlap, their probabilities add. Think of probability as area: non-overlapping regions add up.

Checking an assignment. $S = \{H, T\}$ with $P(H) = p$ and $P(T) = 1 - p$, $0 < p < 1$. Axiom 1: both numbers are in $[0, 1]$. Axiom 2: $P(S) = 1$. Axiom 3: the only disjoint union to check is $\{H\} \cup \{T\} = S$, and $p + (1 - p) = 1$. Valid.

WATCH OUT. If instead $P(T) = 1 - p/2$, Axioms 1 and 2 still look fine, but $P(H) + P(T) = 1 + p/2$, which is not $P(S)$. Axiom 3 fails. When asked "which axiom breaks", check each one separately.

Finite sample spaces. Give each outcome x_i a number $p_i \ge 0$ with $p_1 + \cdots + p_N = 1$. Then $P(E)$ is the sum of the p_i for outcomes in E. Unfair die with $P(1) = P(2) = P(3) = 1/4$ and $P(4) = P(5) = P(6) = 1/12$: $P(\text{even}) = 1/4 + 1/12 + 1/12 = 5/12$.

4.2 What follows from the axioms

FIVE CONSEQUENCES.

- $P(\varnothing) = 0$.
- Disjoint A, B: $P(A \cup B) = P(A) + P(B)$. Same for any finite list of disjoint events.
- **Complement:** $P(E^c) = 1 - P(E)$.
- **Monotone:** if $E \subseteq F$ then $P(E) \le P(F)$.
- **Addition rule:** $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Why the addition rule subtracts. Adding $P(E)$ and $P(F)$ counts the overlap twice. Formally, $E \cup F = E \cup (E^c \cap F)$ is a disjoint union, and $P(E^c \cap F) = P(F) - P(E \cap F)$.

Inclusion–exclusion for three events.

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$- P(E \cap F) - P(E \cap G) - P(F \cap G)$$
$$+ P(E \cap F \cap G)$$

An outcome in all three is counted 3 times in the singles and removed 3 times in the pairs, so it sits at zero and must be added back once. You only need this up to three events.

Example: two overlapping groups. 80% of students have Facebook or Twitter; 60% have Facebook and 25% Twitter. Both? $0.80 = 0.60 + 0.25 - P(\text{both})$, so $P(\text{both}) = 0.05$.

Example: three courses. 100 students: 65 in Prob, 45 in LA, 40 in COMP; 30 in Prob and LA, 25 in Prob and COMP, 20 in LA and COMP; 10 in all three.

1. In at least one: $65 + 45 + 40 - 30 - 25 - 20 + 10 = 85$. In none: 15, so $P(\text{none}) = 0.15$.
2. In exactly one: Prob only $= 65 - 30 - 25 + 10 = 20$ (the $+10$ restores the all-three students, who were subtracted twice); LA only $= 45 - 30 - 20 + 10 = 5$; COMP only $= 40 - 25 - 20 + 10 = 5$. Total 30, $P(\text{exactly one}) = 0.30$.

Example: three sports. 28% watched track, 29% swimming, 19% gymnastics; 14% track and swimming, 12% swimming and gymnastics, 10% track and gymnastics; 8% all three. $P(\text{at least one}) = 0.28 + 0.29 + 0.19 - 0.14 - 0.12 - 0.10 + 0.08 = 0.48$, so $P(\text{none}) = 0.52$.

Exactly / none. "Exactly two of three": subtract the triple from each pair count and add the three results. "Exactly one": each single minus its two pairs plus the triple. "None": compute the union and subtract from 1.

Example: bounds. If $P(A) = 0.4$ and $P(B) = 0.7$ and nothing else is known: since $P(A \cup B) \leq 1$, the addition rule gives $P(A \cap B) \geq 0.4 + 0.7 - 1 = 0.1$; since $A \cap B \subseteq A$, $P(A \cap B) \leq 0.4$.

Example: working backwards. Given $P(A^c \cap B^c) = 0.1$, $P(A \cap B) = 0.3$, $P(B^c) = 0.25$. By De Morgan, $P(A \cup B) = 1 - 0.1 = 0.9$. $P(B) = 0.75$. So $P(A) = 0.9 - 0.75 + 0.3 = 0.45$. Draw the Venn diagram if the algebra feels shaky.

4.3 Equally likely outcomes: probability by counting

Counting formula. If S has N outcomes and each has probability 1/N, then for any event E,

$$P(E) = \frac{\text{outcomes in E}}{N} = \frac{|E|}{|S|}$$

WATCH OUT. Use this only when the outcomes really are equally likely. "Number of heads in two flips" has S = {0, 1, 2}, but 1 head happens two ways. Count at the level of ordered, equally likely outcomes, then group.

Three ways to sample two people from 900 (600 Democrats, 300 Republicans):

- Without replacement, keeping order: $900 \cdot 899$ outcomes, each equally likely.
- Without replacement, ignoring order: $\binom{900}{2}$ outcomes, each equally likely. Fine for any question that only depends on who was picked, since each pair corresponds to exactly two ordered outcomes.
- With replacement: 900^2 ordered outcomes, each equally likely. Keep the order here: an unordered pair {x, y} with $x \neq y$ arises two ways but {x, x} only one way, so unordered outcomes would not be equally likely.

Example: ordered and unordered agree. 10 students, 6 first-years and 4 sophomores; pick 3 at random. P(1 first-year and 2 sophomores)?

Unordered. $\binom{10}{3} = 120$ groups. Favorable: $\binom{6}{1}\binom{4}{2} = 36$. P = 36/120 = 0.3.

Ordered. $10 \cdot 9 \cdot 8 = 720$ sequences. Favorable: position of the first-year (3) × which first-year (6) × two sophomores in order $(4 \cdot 3) = 216$. P = 216/720 = 0.3.

Example: voters. 40 voters: 20 D, 15 R, 5 I. Pick 3 without replacement. P(no Independents) = $\binom{35}{3}/\binom{40}{3} = 6545/9880 \approx 0.6624$. P(one of each) = $(20 \cdot 15 \cdot 5)/9880 \approx 0.1518$.

Example: deer. 300 deer, 100 tagged. Pick 3 without replacement. P(at most 1 tagged) = $[\binom{200}{3} + \binom{100}{1}\binom{200}{2}] / \binom{300}{3} \approx 0.7415$. Answers built from $(2/3)^3$ would treat the draws as independent, which is wrong without replacement.

Example: prizes. 120 guests, 6 prizes drawn at random. P(you win) = 6/120. Counting confirms it: $\binom{119}{5}/\binom{120}{6} = 6/120$. With n guests and k prizes: k/n.

Example: difference divisible by 4. Pick m then n with replacement from 5 to 14 (100 ordered outcomes). $|m - n|$ is a multiple of 4 when the difference is 0 (10 outcomes), 4 (12 outcomes), or 8 (4 outcomes). P = 26/100 = 13/50.

4.4 "At least one": use the complement

COMPLEMENT TRICK. P(at least one) = 1 − P(none). "None" is almost always easier to count.

Birthday problem. n people, 365 equally likely birthdays. P(some two share) = 1 − P(all different), and

$$\text{P(all different)} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365}$$

n	P(at least one shared birthday)
5	0.0271 (so P(no match) = 0.9729)
22	0.4757
23	0.5073, the first n above 50%
100	0.9999997

- **Flags.** One of 27 flags is chosen on each of 3 days. P(the Dutch flag appears at least once) $= 1 - (26/27)^3 \approx 0.1070$.
- **Shoes.** 10 pairs, choose 8 shoes at random. P(no complete pair) $= \binom{10}{8} \cdot 2^8 \,/\, \binom{20}{8} \approx 0.0915$: choose which 8 pairs contribute one shoe, then left or right from each.
- **Lottery repeat.** A 6-of-42 lottery has $N = \binom{42}{6} = 5{,}245{,}786$ combinations. P(the next draw matches a fixed one) $= 1/N$. P(some pair of draws matches among m draws) is a birthday problem with N "birthdays", and crosses 50% at about $m = 2697$ draws. A specific coincidence is rare; some coincidence is not.

4.5 Poker hands

5 cards from 52: $\binom{52}{5} = 2{,}598{,}960$ equally likely hands. A **full house** is three of one denomination plus two of another. Build it in the order you would describe it: the triple's denomination (13), its suits $\binom{4}{3} = 4$, the pair's denomination (12), its suits $\binom{4}{2} = 6$. Favorable $= 13 \cdot 4 \cdot 12 \cdot 6 = 3744$, P ≈ 0.00144.

INTERCHANGEABLE OR NOT?. Triple and pair are different roles, so $13 \cdot 12$. For *two pairs* the two denominations play the same role, so choose them together: $\binom{13}{2}$, not $13 \cdot 12$.

4.6 Check yourself

1. A die has P(1) = 0.3, P(2) = 0.2, P(3) = 0.2, P(4) = 0.1, P(5) = 0.1, P(6) = 0.2. Which axiom fails?

The six numbers add to 1.1, so P(S) ≠ 1: Axiom 2 fails (via Axiom 3). Each number alone satisfies Axiom 1.

2. P(A) = 0.5, P(B) = 0.4, P(A ∩ B) = 0.2. Find P(A ∪ B), P(neither), P(A only).

0.7, then $1 - 0.7 = 0.3$, then $0.5 - 0.2 = 0.3$.

3. Of 200 people, 90 read blog X, 70 read Y, 50 read Z; 30 read X and Y, 20 X and Z, 15 Y and Z; 10 read all three. P(reads none)?

Union $= 90 + 70 + 50 - 30 - 20 - 15 + 10 = 155$. None $= 45$, P $= 0.225$.

4. 5 cards from 52. P(all four aces)?

Four aces plus any one of the other 48 cards: $48/\binom{52}{5} \approx 0.0000185$.

5. A class of 30 has 12 seniors. Pick 4 without replacement. P(exactly 2 seniors)?

$\binom{12}{2}\binom{18}{2}/\binom{30}{4} = 10{,}098/27{,}405 \approx 0.3685$.

6. Roll a fair die 4 times. P(at least one 6)?

$1 - (5/6)^4 \approx 0.5177$.

7. 8 people, 12 equally likely zodiac signs. P(no two share a sign)?

$(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5)/12^8 \approx 0.0464$.

8. $E \subseteq F$, $P(E) = 0.3$, $P(F) = 0.5$. Find $P(F \cap E^c)$.

$F = E \cup (F \cap E^c)$ disjointly, so $0.5 - 0.3 = 0.2$.

9. 80 tablets: 10 defective, 30 working Samsungs, 40 working iPads. Pick 5 at random. P(exactly one defective and at least 3 iPads)?

One defective (10 ways), and the other 4 are either 3 iPads + 1 Samsung ($\binom{40}{3} \cdot 30 = 296{,}400$) or 4 iPads ($\binom{40}{4} = 91{,}390$). Total $10 \cdot 387{,}790$ over $\binom{80}{5} = 24{,}040{,}016$, ≈ 0.1613.

5. Conditional probability

The idea. Learning something changes probabilities. "The probability of E given that F happened" is written P(E | F). Once you know F happened, F becomes your new, smaller sample space. That single picture explains every formula in this section.

5.1 Definition

CONDITIONAL PROBABILITY. If $P(F) > 0$,

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Top: the part of F where E also happens. Bottom: the size of the new sample space F.

WITH EQUALLY LIKELY OUTCOMES.

$$P(E \mid F) = \frac{\text{outcomes in } E \cap F}{\text{outcomes in } F}$$

The denominator is the number of outcomes in F, not in S.

Example: a table.

	In-state	Out-of-state	Total
Statistics major	52	18	70
Other major	15	15	30
Total	67	33	100

Pick a student at random. P(in-state) = 67/100. Told the student is a Statistics major, only that row matters: P(in-state | Statistics) = 52/70. Told "other major": 15/30 = 1/2.

TABLES. Conditioning means "restrict to one row or column and divide by that row or column's total." Count; you rarely need the formula.

Example: two children. Outcomes {BB, BG, GB, GG}, equally likely. A = at least one girl = {BG, GB, GG}. B = both girls = {GG}. P(B | A) = P(GG)/P(A) = (1/4)/(3/4) = 1/3. Reduced sample space: three outcomes have at least one girl, one of them is GG.

WATCH OUT. Instinct says 1/2. That would be right if you were told the *older* child is a girl (reduced space {GB, GG}). "At least one girl" leaves three outcomes. Read the condition exactly as written.

Example: two hearts. Two cards without replacement. B = both hearts, A = at least one heart. Hands with at least one heart: $\binom{52}{2} - \binom{39}{2} = 1326 - 741 = 585$. Hands with two hearts: $\binom{13}{2} = 78$. P(B | A) = 78/585 = 2/15.

Example: divisibility. Pick an integer from 1 to 60. A = divisible by 4 or by 9, B = divisible by 3. Given B, the sample space is the 20 multiples of 3. Of those, A means divisible by 12 or by 9: 5 multiples of 12, 6 multiples of 9, 1 multiple of both (36). $|A \cap B| = 5 + 6 - 1 = 10$, so $P(A \mid B) = 10/20 = 1/2$.

5.2 The multiplication rule: probability of a chain

MULTIPLICATION RULE. Rearranging the definition: $P(E \cap F) = P(F) \cdot P(E \mid F)$. For a chain,

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2)$$

and so on for any number of events. First E_1 happens; then E_2, given E_1; then E_3, given both.

- **Two draws.** 50 students, 20 love probability. Pick 2 without replacement. P(both love it) = (20/50)(19/49) ≈ 0.1551. Same as $\binom{20}{2}/\binom{50}{2}$.
- **Two exams.** P(pass first) = 0.9, P(pass second | passed first) = 0.8. P(pass both) = 0.72.
- **Four draws.** 8 red and 6 blue candies drawn one at a time without replacement. P(blue, red, red, blue) = (6/14)(8/13)(7/12)(5/11) ≈ 0.0699. Each fraction is what is left in the box at that moment.

WITHOUT REPLACEMENT. The multiplication rule is usually fastest: one fraction per draw, updating top and bottom. The birthday product $(364/365)(363/365) \cdot \cdot \cdot$ is the same idea.

5.3 Splitting into cases

TWO-CASE FORMULA. For any events E and F,

$$P(E) = P(E \cap F) + P(E \cap F^c) = P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)$$

Either F happens or it does not, and the two cases do not overlap.

Example: emailing a student. Two course sections, each chosen with probability 1/2. Section 1 is 30% actuarial students, Section 2 is 50%. An actuarial student replies with probability 0.30 in Section 1 and 0.80 in Section 2; other students never reply.

(1) P(Section 1 and a reply): a reply needs an actuarial student, so multiply along the chain: (1/2)(0.30)(0.30) = 0.045.

(2) P(reply): add the two sections. Section 2 gives (1/2)(0.50)(0.80) = 0.20. Total 0.045 + 0.20 = 0.245.

Example: a screening test. 1% of babies have a condition. The test is positive for 90% of affected babies and for 5% of unaffected babies. P(positive) = (0.01)(0.90) + (0.99)(0.05) = 0.0585.

5.4 Conditioning on the wrong thing

Two famous mistakes.

- Multiplying P(A) · P(B) for two events only works if they are independent. Two crib deaths in one family are not independent (shared genes and home), so 1/8500 × 1/8500 was the wrong calculation in a real court case.
- P(partner is the killer | woman was abused) is small, about 1 in 2500. But once the woman *has been* murdered, the relevant number is P(partner is the killer | abused *and* murdered). Out of 100,000 abused women, about 40 are killed by the partner and about 5 by someone else, so that probability is about 40/45 ≈ 89%. The extra condition changes everything.

5.5 Check yourself

1. Roll two dice. E = first die is 3, F = sum is 8. Find P(E | F).

F = {(2,6),(3,5),(4,4),(5,3),(6,2)}. One of the five has a 3 first: 1/5.

2. Pick an integer from 1 to 100. Given it is even, P(divisible by 5)?

50 evens; even and divisible by 5 means divisible by 10: 10 of them. 10/50 = 0.2.

3. A box has 5 good and 3 bad bulbs. Draw 3 without replacement. P(good, good, bad)?

(5/8)(4/7)(3/6) = 5/28 ≈ 0.1786.

4. P(A) = 0.6, P(B | A) = 0.5, P(B | A^c) = 0.2. Find P(A ∩ B) and P(B).

P(A ∩ B) = 0.30. P(B) = 0.30 + (0.4)(0.2) = 0.38.

5. Three cards from a deck. P(all spades)?

(13/52)(12/51)(11/50) ≈ 0.0129.

6. Two dice; you are told at least one shows a 6. P(sum is 8)?

11 outcomes contain a 6. Sum 8 among them: (2,6), (6,2). P = 2/11.

7. Can P(E | F) be larger than, smaller than, or equal to P(E)?

All three. E may take up a bigger, smaller, or equal share of F compared with its share of S. "Equal" is what independence means (section 6).

8. A flag is chosen from 27 on each of three days, repeats allowed. Given Wednesday's flag is Dutch, P(the Dutch flag appears exactly once)?

Thursday and Friday must both be other flags: $(26/27)^2 \approx 0.9273$.

6. Just in case: total probability, Bayes, independence

These were taught in the two classes before the exam but are officially on Midterm 2. They are short extensions of section 5. Learn 6.1 at minimum.

6.1 Law of total probability

Average over the cases. If F_1, ..., F_n are disjoint and cover S, then

$$P(E) = P(E \mid F_1)\,P(F_1) + \cdots + P(E \mid F_n)\,P(F_n)$$

Each case's conditional probability, weighted by how likely that case is.

- **Triage.** 15% of patients are critical (50% die), 45% serious (20% die), 40% stable (2% die). P(die) = 0.075 + 0.09 + 0.008 = 0.173. P(survive) = 0.827.
- **Voting.** 45% independents (30% vote), 30% Democrats (50%), 25% Republicans (60%). P(voted) = 0.135 + 0.15 + 0.15 = 0.435.
- **Spam.** 85% of email is spam. "Free money" appears in 10% of spam and 1% of the rest. P(phrase) = 0.085 + 0.0015 = 0.0865.
- **Mammogram.** P(cancer) = 0.008, P(+ | cancer) = 0.9, P(+ | no cancer) = 0.07. P(+) = 0.0072 + 0.0694 = 0.0766.

6.2 Bayes' formula: flipping the condition

Bayes.

$$P(F \mid E) = \frac{P(E \mid F)\,P(F)}{P(E)} = \frac{P(E \mid F)\,P(F)}{P(E \mid F)\,P(F) + P(E \mid F^c)\,P(F^c)}$$

Top: the case you care about. Bottom: the total probability of the evidence.

- **Mammogram.** P(cancer | +) = 0.0072/0.0766 ≈ 0.094. Only 9%, because the false positives from the large healthy group outnumber the true positives.
- **Screening test.** P(condition) = 0.01, positive for 90% of affected and 5% of unaffected. P(condition | +) = 0.009/0.0585 ≈ 0.154.
- **Rare disease, near-perfect test.** P(disease) = 0.0001, sensitivity 0.999, false-positive rate 0.0001. P(disease | +) = 0.0000999/(0.0000999 + 0.00009999) ≈ 0.50. A positive result is a coin flip.
- **Noisy channel.** A 0 or 1 is sent with equal probability. A sent 0 flips with probability 0.05, a sent 1 with 0.10. P(sent 1 | received 1) = (0.5)(0.9)/[(0.5)(0.9) + (0.5)(0.05)] = 0.45/0.475 ≈ 0.947.

Rare condition + imperfect test. Compute both pieces of the denominator and compare them. When the base rate is tiny, the "healthy but positive" piece usually wins.

Chained milestones. $P(A_2 \mid A_1) = 0.8$, $P(A_2 \mid A_1^c) = 0.3$, and the same for A_3 given A_2; only the most recent milestone matters. Then $P(A_3 \mid A_1) = (0.8)(0.8) + (0.3)(0.2) = 0.70$, $P(A_3 \mid A_1^c) = (0.8)(0.3) + (0.3)(0.7) = 0.45$, and with $P(A_1) = 0.75$, $P(A_3) = (0.75)(0.70) + (0.25)(0.45) = 0.6375$.

6.3 Independence

INDEPENDENT EVENTS. E and F are independent if $P(E \cap F) = P(E)\,P(F)$. Equivalently (when both have positive probability) $P(E \mid F) = P(E)$: knowing F does not change the chance of E.

- Two coin tosses, E = first is H, F = second is T: $P(E \cap F) = 1/4 = (1/2)(1/2)$. Independent.
- One card, E = club, F = a 6: $P(E \cap F) = 1/52 = (1/4)(1/13)$. Independent.
- Two dice, F = first die is 4. E_1 = sum is 6: $P(E_1 \cap F) = 1/36$ but $P(E_1)P(F) = (5/36)(1/6)$. Not independent. E_2 = sum is 7: $P(E_2 \cap F) = 1/36 = (6/36)(1/6)$. Independent.
- If E and F are independent, so are E and F^c.
- Three events are independent if every pair multiplies and $P(E \cap F \cap G) = P(E)P(F)P(G)$.

INDEPENDENT IS NOT DISJOINT. Disjoint events with positive probability are never independent: if one happens, the other cannot, so knowing one changes the other's probability to zero.

Four friends, one flat tire. Each independently names which of four tires went flat. Two pick uniformly (1/4 each). One favors left tires (1/3 each, 1/6 for right); one favors right (the reverse). P(all four name the same tire) = 4 tires $\times$ $(1/4)(1/4)(1/3)(1/6) = 1/72$, since for every tire the two biased friends contribute $1/3 \cdot 1/6$ either way.

5 before 7. Roll two dice repeatedly. P(a sum of 5 appears before a sum of 7) $= P(5)/(P(5) + P(7)) = (4/36)/(10/36) = 0.4$. Rolls that are neither are ignored; only the first decisive roll counts.

7. Exam strategy

7.1 Four questions before you compute

1. **What is the sample space, and are outcomes equally likely?** If you use |E|/|S|, count both at the same level (ordered with ordered, unordered with unordered).
2. **Does order matter?** Swapping two items: new outcome or not? Permutation or combination.
3. **With or without replacement?** With: powers like 27^3 or $(26/27)^3$. Without: shrinking fractions like $(20/50)(19/49)$, or binomial coefficients.
4. **Is the complement easier?** "At least one", "no two the same", "not both": compute the opposite and subtract from 1.

7.2 Multiple-choice tactics

- **Interval answers:** compute to 4 decimals, then check whether the endpoints are included. (0.8, 0.85) excludes both; (0.1, 0.2] includes 0.2.
- **Formula answers:** check the pieces. Right denominator? Groups labeled or not? Are all the cases added ("at most 1" = 0 + 1)?
- **Sanity limits:** probabilities are in [0, 1]; $P(E \cap F) \leq \min(P(E), P(F))$; $P(E \cup F) \geq \max(P(E), P(F))$; an added condition can only lower a probability.
- **Set questions:** test concrete outcomes or write the eight-row yes/no table.
- **Time:** about 3.5 minutes per question. Every question is worth the same; skip and come back.

7.3 The traps

Trap	Correct thinking
Dividing by 3! when splitting into three groups with different jobs	Named groups: plain multinomial, no 3!.
Using $(2/3)^3$-style answers for draws without replacement	Draws without replacement are not independent: binomials or shrinking fractions.
Forgetting $+ P(E \cap F \cap G)$	Singles − pairs + triple.
"At least one girl" giving 1/2	Reduced space has 3 outcomes: 1/3.
Dividing by |S| in a conditional	Given F, divide by the number of outcomes in F.
Writing $(E \cup F)^c$ as $E^c \cup F^c$	Complement swaps the operation: $E^c \cap F^c$.
Confusing disjoint with independent	Disjoint: cannot both happen. Independent: one does not affect the other.
Treating {0, 1, 2} heads as equally likely	Go back to {HH, HT, TH, TT}.
Computing 365! on the calculator	Multiply the fractions one at a time.

Mixing up $(1 - a/n)^n$ and a fixed base	Moving base $\to e^{-a}$. Fixed base below $1 \to 0$.

8. Practice exams

Part A is the practice midterm posted for the course, with worked answers. Part B is a new set in the same style. Do each part in 50 minutes with only a calculator, then read every solution, including the ones you got right. The real exam will not just be variants of these, so focus on the reasoning.

8.1 Part A: the posted practice midterm

A1. $\Sigma_{n=0}^{\infty}\ (-2)^n/n!$ is in which interval? (0.1, 0.2], (0.2, 0.3], (0.3, 0.4], (0.4, 0.5], (0.5, 0.6]

$e^{-2} \approx 0.1353$. **(A) (0.1, 0.2].**

A2. Olympics viewers: 28% track, 29% swimming, 19% gymnastics; 14% track and swimming, 12% swimming and gymnastics, 10% track and gymnastics; 8% all three. P(a random viewer watched none)?

P(at least one) $= 0.28 + 0.29 + 0.19 - 0.14 - 0.12 - 0.10 + 0.08 = 0.48$. P(none) $= 1 - 0.48 = 0.52$. **(E) (0.5, 0.6].**

A3. 300 deer, 100 with trackers. Pick 3 without replacement. P(at most 1 tracked deer) equals which expression?

Equally likely unordered samples: $\binom{300}{3}$. Favorable: 0 tracked, $\binom{200}{3}$; or 1 tracked, $\binom{100}{1}\binom{200}{2}$. **(D)** [$\binom{200}{3} + \binom{100}{1}\binom{200}{2}$] / $\binom{300}{3} \approx 0.7415$. Options with $(2/3)^3$ treat the draws as independent, which is wrong without replacement; option E forgets the $\binom{200}{2}$ factor; option B counts tracked deer only.

A4. Gandalf splits 90 soldiers into three groups of 30: gate, patrol, guard Frodo. Number of ways?

Three labeled groups (different jobs): **(A)** $\binom{90}{30,\ 30,\ 30} = 90!/(30!)^3$. No 3! anywhere.

A5. 4 dogs and 4 cats sit at random in 8 chairs, 4 per side. P(all dogs on one side)?

Total seatings 8!. Favorable: choose the dogs' side (2), seat dogs there (4!), seat cats on the other side (4!). **(C)** $2(4!)^2/8! \approx 0.0286$.

A6. With A, B, C all overlapping, $[(A \cup B) \cap (A \cup C) \cap (B^c \cap C^c)]$ equals?

$B^c \cap C^c$ says neither B nor C happens; then $A \cup B$ forces A and $A \cup C$ forces A. **(B)** $A \cap B^c \cap C^c$.

A7. $(1/3)^2(1/4)^2 + (1/3)^3(1/4)^4 + (1/3)^4(1/4)^6 + \cdots + (1/3)^{n+1}(1/4)^{2n} + \cdots$

Term n is $(1/3)(1/3)^n(1/16)^n = (1/3)(1/48)^n$. Sum over $n \geq 1$: $(1/3) \cdot (1/48)/(1 - 1/48) = (1/3)(1/47) = 1/141 \approx 0.0071$. None of the printed intervals contain this value (they all start at 0.1), so the options on the posted practice exam appear to have a typo. The method is what matters: factor out the common part, find the ratio, use $a/(1 - a)$.

A8. Class of 10 including Brad and John. Number of subsets not containing both Brad and John?

$2^{10} - 2^8 = 1024 - 256 = 768$. **(D) (600, 800].**

A9. 9 students (3 seniors, 3 juniors, 3 sophomores) randomly split into three labeled groups of 3 (TER, GC, NE). P(each group has one of each class)?

Total labeled splits: $9!/(3!)^3 = 1680$. Favorable: send the 3 seniors to the 3 groups one each (3! ways), same for juniors (3!) and sophomores (3!): 216. $P = 216/1680 \approx 0.1286$. **(E) (0.1, 1.0].**

A10. $\lim_{n\to\infty} (1 - 3/n)^n$

$e^{-3} \approx 0.0498$. **(B) (0.03, 0.07].**

A11. 5 people; P(no two share a birthday)?

$(365 \cdot 364 \cdot 363 \cdot 362 \cdot 361)/365^5 \approx 0.9729$. **(E) (0.95, 1).**

A12. Invite 5 of 8 friends; 2 feuding friends will not both attend. Number of possible groups?

$\binom{8}{5} - \binom{6}{3} = 56 - 20 = 36$. **(C) (30, 45].**

8.2 Part B: a second exam

B1. $\Sigma_{n=0}^{\infty} (-1.5)^n/n!$ lies in: (0.1, 0.2], (0.2, 0.3], (0.3, 0.4], (0.4, 0.5].

$e^{-1.5} \approx 0.2231 \to (0.2, 0.3]$.

B2. $\Sigma_{n=2}^{\infty} (0.6)^n$ lies in: (0.7, 0.8], (0.8, 0.9], (0.9, 1.0].

Geometric starting at $n = 2$: $0.6^2/(1 - 0.6) = 0.36/0.4 = 0.9 \to (0.8, 0.9]$.

B3. A license plate has 3 letters followed by 4 digits, no letter repeated and no digit repeated. A plate is chosen at random. P(it contains the letter A and the digit 1)? Round to 4 decimals.

Letters: strings with an A among 3 distinct letters = $3 \cdot 25 \cdot 24 = 1800$ out of $26 \cdot 25 \cdot 24 = 15{,}600$, so 3/26. Digits: 4/10 by the same logic. The letter part and digit part are chosen independently, so multiply: $(3/26)(4/10) \approx 0.0462$.

B4. $(A \cap B) \cup (A \cap B^c)$ equals: A, B, $A \cup B$, $A \cap B$, or $\varnothing$?

Distributive law: $A \cap (B \cup B^c) = A \cap S = A$. In words: "A with B" or "A without B" is just A.

B5. How many distinct arrangements of the letters in PEPPER?

6 letters: P×3, E×2, R×1. $6!/(3!\,2!\,1!) = 60$.

B6. A committee of 4 is chosen at random from 6 women and 5 men. P(at least 2 women) lies in: (0.7, 0.75), (0.75, 0.8), (0.8, 0.85), (0.85, 0.9).

Total $\binom{11}{4} = 330$. Complement: 0 women $\binom{5}{4} = 5$; 1 woman $6 \cdot \binom{5}{3} = 60$. So $330 - 65 = 265$ favorable, $P = 265/330 \approx 0.8030 \rightarrow (0.8, 0.85)$.

B7. P(A) = 0.5, P(B) = 0.4, P(C) = 0.3, P(A ∩ B) = 0.2, P(A ∩ C) = 0.15, P(B ∩ C) = 0.1, P(A ∩ B ∩ C) = 0.05. Find P(exactly one of the three occurs).

A only $= 0.5 - 0.2 - 0.15 + 0.05 = 0.20$; B only $= 0.4 - 0.2 - 0.1 + 0.05 = 0.15$; C only $= 0.3 - 0.15 - 0.1 + 0.05 = 0.10$. Total 0.45. (Also: union $= 1.2 - 0.45 + 0.05 = 0.80$, so P(none) = 0.20.)

B8. 4 people, birth months equally likely among 12. P(at least two share a birth month) lies in: (0.3, 0.4], (0.4, 0.5], (0.5, 0.6].

$1 - (12 \cdot 11 \cdot 10 \cdot 9)/12^4 = 1 - 11{,}880/20{,}736 \approx 1 - 0.5729 = 0.4271 \rightarrow (0.4, 0.5]$.

B9. 12 people are split into three teams of 4 to play the same game; the teams have no names or roles. Number of ways: $12!/(4!)^3$, $12!/((4!)^3\,3!)$, $3! \cdot 12!/(4!)^3$, or $\binom{12}{4}$?

Unlabeled equal-size groups: $12!/((4!)^3\,3!) = 5775$. The labeled count 34,650 overcounts each split 3! times.

B10. Two fair dice. Given the sum is at least 10, P(both dice show the same number)?

Sum ≥ 10: (4,6),(5,5),(6,4),(5,6),(6,5),(6,6), six outcomes. Doubles among them: (5,5),(6,6). $P = 2/6 = 1/3$.

B11. An urn has 7 red and 5 blue balls. Draw 3 without replacement. P(red, then blue, then red) to 4 decimals.

(7/12)(5/11)(6/10) = 210/1320 ≈ 0.1591.

B12. S = {a, b, c} with P({a}) = 0.5, P({b}) = 0.3, P({c}) = 0.3. Which axiom is violated?

Each number is in [0, 1] (Axiom 1 fine), but by Axiom 3 P(S) = 0.5 + 0.3 + 0.3 = 1.1 ≠ 1, so Axiom 2 is violated (the assignment is inconsistent with Axioms 2 and 3 together).

B13. Five cards from 52. P(two pairs: two cards of one denomination, two of another, and a fifth of a third denomination) equals which expression? (i) $13 \cdot 12 \cdot \binom{4}{2}^2 \cdot 44 / \binom{52}{5}$; (ii) $\binom{13}{2}\binom{4}{2}^2 \cdot 44 / \binom{52}{5}$; (iii) $\binom{13}{2}\binom{4}{2}^2 \cdot 48 / \binom{52}{5}$.

(ii). The two pair denominations are interchangeable, so choose them together: $\binom{13}{2}$ ($13 \cdot 12$ double counts). Suits: $\binom{4}{2}$ each. Fifth card: any of the 44 cards not of those two denominations (48 would allow a full house). Value ≈ 0.0475.

B14. Toss a coin 3 times and record only the number of heads. How many outcomes are in S, and are they equally likely?

S = {0, 1, 2, 3}, four outcomes, not equally likely: P = 1/8, 3/8, 3/8, 1/8, because 1 head can occur in 3 of the 8 ordered sequences.

B15. Coin tossed until the first tail; E_k = at least k tosses needed. List the outcomes in $E_3 \cap E_5^c$.

At least 3 tosses but fewer than 5: exactly 3 or 4 tosses: {HHT, HHHT}.

9. One-page summary

Read this the night before and again right before the exam.

Series and limits. $1 + a + \cdots + a^n = (1 - a^{n+1})/(1 - a)$. Infinite ($|a| < 1$): $1/(1 - a)$; from a: $a/(1 - a)$. $\Sigma\, x^n/n! = e^x$. $(1 - a/n)^n \to e^{-a}$. Fixed base < 1 to the $n \to 0$. $0! = 1$. $n(n-1) \cdots (n-r+1) = n!/(n-r)!$.

Counting. Stages multiply: $n_1 n_2 \cdots n_r$. Subsets: 2^n. Arrange n: n!. Choose r: $\binom{N}{R} = n!/(r!(n-r)!)$. Labeled groups: $n!/(n_1! \cdots n_r!)$; unlabeled equal groups: also $\div$ (number of groups)!. Repeated letters: $n!/(n_1! \cdots)$. Binomial: $(x+y)^n = \Sigma \binom{N}{R} x^r y^{n-r}$. Feud: all − both. Together: glue a block. Not adjacent: choose gaps.

Events. S = all outcomes (depends on what you record). Event = subset. $\cup$ or, $\cap$ and, ^{c} not, $\varnothing$ impossible, disjoint = $\cap$ is $\varnothing$, $\subseteq$ implies. Distributive: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$; $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$. De Morgan: $(E \cup F)^c = E^c \cap F^c$; $(E \cap F)^c = E^c \cup F^c$. $E \cup F = E \cup (E^c \cap F)$. Exactly two of A,B,C: union of three "two yes one no" intersections. $A \cap B \subset A$ so $P(A \cap B) \leq P(A)$.

Axioms. (1) $0 \leq P(E) \leq 1$. (2) $P(S) = 1$. (3) disjoint E_i: $P(\cup E_i) = \Sigma P(E_i)$. Then: $P(\varnothing) = 0$; $P(E^c) = 1 - P(E)$; $E \subseteq F \Rightarrow P(E) \leq P(F)$; $P(E \cup F) = P(E) + P(F) - P(E \cap F)$; three events: singles − pairs + triple. $P(A \cap B) \geq P(A) + P(B) - 1$. Equally likely: $P(E) = |E|/|S|$ (same level of counting top and bottom).

Classic counts. Birthday: P(all different) $= 365 \cdot 364 \cdots (365-n+1)/365^n$; 23 people → just over 50%. Without replacement: $\binom{A}{K}\binom{B}{M}/\binom{A+B}{K+M}$. With replacement, "at least one" $= 1 - (1 - p)^n$. Full house: $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}/\binom{52}{5} \approx 0.00144$. iPads: k/n. Bridge: $52!/(13!)^4$.

Conditional. $P(E|F) = P(E \cap F)/P(F) = |E \cap F|/|F|$ (equally likely). F is the new sample space. Multiplication: $P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)$. Without replacement: chain of shrinking fractions. Two kids, at least one girl: P(both girls) = 1/3. Two cases: $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$.

Total probability, Bayes. Total probability over a partition: $P(E) = \Sigma\, P(E|F_i)P(F_i)$. Bayes: $P(F|E) = P(E|F)P(F)/P(E)$. Mammogram: 9%. Independent: $P(E \cap F) = P(E)P(F)$; disjoint $\neq$ independent. 5 before 7: 4/10.

Numbers to have ready. $e^{-1} = 0.3679$, $e^{-2} = 0.1353$, $e^{-3} = 0.0498$. $\binom{52}{5} = 2{,}598{,}960$. $\binom{52}{2} = 1326$. $2^{10} = 1024$. 5 people no shared birthday: 0.9729. 8! = 40,320. 10! = 3,628,800.

Exam. 50 minutes, 12–15 multiple choice, closed book, calculator required, round to 4 decimals. Check whether interval endpoints are included.

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