STOR 435 Midterm 1 Study Guide

Introduction to Probability · UNC Fall 2026 · Section 002 (Bhamidi) · Covers Lectures 1–4 and Homework 1–4

0. Exam logistics and how to use this guide

When. Tuesday, September 15, 2026, 12:30–1:45 p.m. (the regular class period), Hanes Hall 120. Arrive a few minutes early.

Format. 18–20 multiple-choice questions on a Gradescope bubble sheet. Only the bubble sheet is graded; scratch work is not. Worth 15% of the course grade.

Conditions. In person, closed book, closed notes, *no formula sheet.* Bring pens or pencils and a non-networked scientific calculator. No phones, laptops or smart watches.

What is in scope

The information sheet says the exam covers "material from Lectures 1–4, together with the corresponding homework, worksheets, and prerequisite ideas used in those lectures." You should "be able to recognize which tools apply, carry out the calculation, and interpret the answer." Homework 1 is explicitly part of the syllabus, "especially Problems 1–9 and 14," because they cover prerequisite material that does not appear on the slides.

Unit	Source	What you must be able to do
Prerequisites	HW1 Q1–9, Q14	Set difference; summation notation and finite sums; binomial coefficients as coefficients of $(x+y)^n$; finite and infinite geometric series; the exponential series $\sum x^k/k!$; normalising a density so it integrates to 1.
L01 Counting	Lecture 1, HW2, bank Q1–10, 22–25	Product rule and its r-stage generalisation; permutations $n!$ and $P(n,r)$; combinations $\binom{n}{r}$; number of subsets 2^n; binomial and multinomial theorems; multinomial coefficients as labelled groups and as arrangements with repeated objects; constrained counts by cases or complement; turning a count into a probability.
L02 Sample spaces and events	Lecture 2, HW3, bank Q11–21	Choose S at the right granularity; classify S as finite / countably infinite / continuous; events as subsets; union, intersection, complement, difference, disjointness, inclusion; commutative, associative, distributive laws; De Morgan; translating words ↔ symbols ("exactly one", "at least two", "none").
L03 Axioms	Lecture 3, HW3 Q15–16, HW4 Q2–11, bank Q26–35	The three axioms; $\mathbb{P}(\emptyset)=0$, finite additivity, complement rule, monotonicity; point-mass models and auditing whether numbers form a valid model; equally likely outcomes $\mathbb{P}(E)=\lvert E\rvert/\lvert S\rvert$; sampling with vs without replacement; the decomposition $A=(A\cap B)\,\dot\cup\,(A\cap B^c)$; two- and three-event inclusion–exclusion; "exactly one" and "none" of three events; bounds on unions and intersections; birthday problem; poker hands.

L04 Conditional probability	Lecture 4, HW4 Q12–15, bank Q36–45	$\mathbb{P}(E \mid F) = \mathbb{P}(E \cap F)/\mathbb{P}(F)$ as restriction plus renormalisation; reading conditionals from a two-way table; $\mathbb{P}(E \mid F) \neq \mathbb{P}(F \mid E)$; multiplication rule; chain rule along a sequence; with vs without replacement as conditional factors; multiply along a tree path, add across disjoint paths.

Not on this exam. The information sheet explicitly excludes "the Extra Reading, formal independence, Bayes' rule, and material from Lecture 5 onward." So: no total-probability formula as a named theorem, no Bayes, no independence definition, no random variables. The extra-reading pages in each deck (hash collisions, Bonferroni, AI benchmarks, the brain-as-betting-engine pages) are out too. You may still need to *multiply along a tree and add disjoint routes*, because that was set up in Lecture 4.

How to study with this guide

- **Each lecture chapter** follows the slide deck in order, expands the terse slides, works every board-work problem the slides posed, and ends with "Check yourself" questions with hidden answers.
- **Boxes:** blue "Key idea" boxes hold definitions and formulas quoted from the slides (memorise these: there is no formula sheet). Orange "Watch out" boxes flag the mistakes that cost points on the homework and the practice bank. Green "Exam tip" boxes say how a multiple-choice question on that topic tends to be phrased.
- **Homework problems** are reworked inside the relevant chapter, including the two that were marked wrong on the graded copies (HW2 Q9 and HW3 Q9). Those are exactly the kind of slip a multiple-choice exam punishes.
- **The practice exam** at the end is a mixed set in the style of the practice bank: multiple-choice, all four lectures shuffled, with the "which correction is appropriate" audit style the instructor likes.
- **The cram sheet** at the very end is one screen of formulas and traps. Read it last.

Exam tip. The instructor's own advice on the information sheet: practise "moving between words, set notation, diagrams or trees, and numerical calculations," and check "basic consistency: probabilities must lie between 0 and 1, and counting answers should match the structure of the experiment." Many bank questions are precisely "a student wrote X; which correction is appropriate?" Expect to diagnose, not just compute. Also expect answers given as *intervals* ("N lies in (30, 45]"), which is how the previous practice midterm questions were written.

Extra office hours. Monday September 14, 4:30–6:00 p.m. and Tuesday September 15, 8:30–9:30 a.m., both on Zoom (link and passcode on the information sheet). Regular Tuesday afternoon office hours on September 15 are cancelled.

HW1 1. Prerequisites the slides assume

Homework 1 is on the exam syllabus "especially Problems 1–9 and 14" because Lectures 1–4 use these tools without stopping to teach them. Each problem below is reworked with the reasoning the course wants, and the "why this matters" line from the homework is kept because it tells you where the tool reappears.

1.1 Set difference

Set difference. $A \setminus B$ (also written $A - B$) is the set of elements of A that are *not* in B. In event language, $E \setminus F = E \cap F^c$: "$E$ occurs but F does not."

HW1 Q2. $A = \{1,2,3,4\}$, $B = \{3,4,5\}$. Then $A \setminus B = \{1,2\}$: remove from A whatever also appears in B. Note the 5 in B is irrelevant. Lecture 2 uses $E \setminus F$ to translate phrases like "the second ball is black but the third is not."

1.2 Summation notation and finite sums

HW1 Q3. $\sum_{k=2}^{4}(2k-1)$ expands term by term: $k=2$ gives 3, $k=3$ gives 5, $k=4$ gives 7, so it equals $3+5+7$. The index runs from the lower limit to the upper limit inclusive.

HW1 Q4. $\sum_{j=1}^{10}(3j-2) = 3\sum_{j=1}^{10} j - \sum_{j=1}^{10} 2 = 3 \cdot 55 - 20 = 145$, using $\sum_{j=1}^{n} j = n(n+1)/2$ and the fact that a constant summed n times is n times the constant.

HW1 Q5 (double sum). $\sum_{i=1}^{3}\sum_{j=0}^{2}(i+2j)$. Do the inner sum for fixed i: $\sum_{j=0}^{2}(i+2j) = 3i + 2(0+1+2) = 3i+6$. Then the outer sum: $\sum_{i=1}^{3}(3i+6) = 3 \cdot 6 + 18 = 36$. Lecture 3 sums point masses over the outcomes in an event; later lectures sum over one coordinate to get a marginal.

Exam tip. A summation question on this exam will most likely be disguised as "add the point masses of the outcomes in E" (Lecture 3) or "add the probabilities along the disjoint tree paths" (Lecture 4). The mechanics are the same: expand, group, add.

1.3 Binomial coefficient as an algebraic coefficient

Binomial theorem.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

The coefficient of $x^r y^{n-r}$ is $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$: choose which r of the n factors contribute an x.

HW1 Q6. Coefficient of x^3y^4 in $(x+y)^7$ is $\binom{7}{3} = \binom{7}{4} = 35$.

Watch out: scalars inside the binomial. If the binomial is $(2x-y)^8$ (HW2 Q9) the coefficient of x^3y^5 is $\binom{8}{3}\,2^3\,(-1)^5 = 56 \cdot 8 \cdot (-1) = -448$, *not* 56. The graded HW2 copy answered 56 and lost the point. Practice-bank Q6 and Q23 are the same trap: $(3u-2v)^7$ gives $\binom{7}{4}3^4(-2)^3 = -22680$ for u^4v^3. Always multiply the counting coefficient by the powers of the scalars.

1.4 Geometric series

FINITE AND INFINITE GEOMETRIC SUMS.

$$\sum_{k=0}^{n} ar^k = a\,\frac{1-r^{n+1}}{1-r} \quad (r \neq 1), \qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad (|r|<1).$$

If $|r| \geq 1$ the infinite series diverges.

HW1 Q7. $\sum_{k=0}^{5} 3\left(\frac{1}{2}\right)^k = 3 \cdot \dfrac{1-(1/2)^6}{1-1/2} = 3 \cdot 2 \cdot \dfrac{63}{64} = \dfrac{378}{64} = 5.90625$.

HW1 Q8. $\sum_{k=0}^{\infty} 4\left(\frac{1}{3}\right)^k$ converges because $|1/3|<1$, to $\dfrac{4}{1-1/3} = \dfrac{4}{2/3} = 6$. "Infinitely many positive terms" does not by itself force divergence. Lecture 6 will use this to check that infinitely many probabilities add to 1.

1.5 Exponential series and limit

TWO WAYS TO REACH e^x.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

HW1 Q9. $\sum_{k=0}^{\infty} \dfrac{(-2)^k}{k!} = e^{-2} \approx 0.1353$.

HW1 Q10. $\lim_{n\to\infty}(1+5/n)^n = e^5$. (Q10 is not in the "especially" list, but it is one line.)

1.6 Normalising a density

HW1 Q14. For which c does $f(x) = cx(1-x)$ on $[0,1]$ (and 0 elsewhere) integrate to 1? Compute $\int_0^1 x(1-x)\,dx = \int_0^1 (x-x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $c \cdot \frac{1}{6} = 1$ and $c = 6$. The logic is identical to the discrete case in Lecture 3: point masses (or a density) must add (integrate) to exactly 1 before they define a model. HW3 Q16 is the discrete twin: $\mathbb{P}(G) = c, \mathbb{P}(Y) = 2c, \mathbb{P}(O) = 3c, \mathbb{P}(R) = 4c$ forces $10c = 1$.

Extra context beyond the homework: Q11–13 (definite integral $= 12$, Riemann sum $= \int_0^1 x^2\,dx$, double integral $= 0.5$) and Q15 (committee with Ana and Bo forced in: $\binom{6}{3} = 20$) are not in the "especially" list; Q15 is really a Lecture 1 problem and reappears in §2.

1.7 Check yourself

1. Let $E = \{a,b,c,d\}$ and $F = \{c,d,e\}$ inside $S = \{a,b,c,d,e,f\}$. Find $E \setminus F$, $F \setminus E$, and $(E \setminus F)^c$. [HW1]

$E \setminus F = \{a,b\}$; $F \setminus E = \{e\}$; $(E \setminus F)^c = S \setminus \{a,b\} = \{c,d,e,f\}$. Practice-bank Q16 is exactly this shape.

2. Compute $\sum_{k=1}^{4} \sum_{j=1}^{2} (k \cdot j)$. [HW1]

Inner sum for fixed k: $k(1+2) = 3k$. Outer: $3(1+2+3+4) = 30$.

3. What is the coefficient of x^2y^3 in $(x - 3y)^5$? [HW1 · L01]

$\binom{5}{2} \cdot 1^2 \cdot (-3)^3 = 10 \cdot (-27) = -270$. Forgetting the scalar gives the wrong answer 10.

4. Does $\sum_{k=0}^{\infty} 5\,(1.2)^k$ converge? What about $\sum_{k=0}^{\infty} 5\,(0.2)^k$? [HW1]

The first diverges ($\lvert r \rvert = 1.2 \geq 1$). The second converges to $5/(1-0.2) = 6.25$.

5. A model assigns $\mathbb{P}(k) = c\,(1/2)^k$ for $k = 0, 1, 2, \ldots$. What must c be? [HW1 · L03]

$\sum_{k \geq 0} c(1/2)^k = c \cdot \frac{1}{1-1/2} = 2c = 1$, so $c = 1/2$.

L01 2. Counting and combinatorics

Lecture 1 spends its first half on motivation (mammograms, Sally Clark, the O. J. Simpson trial, the course roadmap) and its second half on the counting toolkit. The motivation is context: the exam-relevant content starts at the "Basic principle of counting" slide. The slides' own summary of the goals: "describe what one outcome looks like and decide whether order or repetition matters; use the multiplication principle and distinguish permutations from combinations; count the outcomes in a finite set needed for a probability calculation; check a counting answer by working out a small case."

2.1 Why the motivation slides still matter

Three stories from the deck recur in later lectures and in the instructor's phrasing. Know what each one is *about*, not the numbers.

- **Mammogram (Gigerenzer).** Base rate 0.8%, sensitivity 90%, false-positive rate 7%. Doctors guessed 90% or 50–80% for $\mathbb{P}(\text{cancer} \mid +)$; the correct value is about 9%. Lesson: "Probability separates what a test reports from what we want to know." The calculation itself is Bayes and is *not* on this exam.
- **Sally Clark.** A paediatrician multiplied $1/8500 \times 1/8500$ for two SIDS deaths. The deck's question: "Why should the two deaths be independent of each other?" Lesson: multiplying probabilities assumes independence, which must be justified.
- **O. J. Simpson / Dershowitz.** "Fewer than 1 homicide per 2500 incidents of abuse" was presented as evidence that battery is not a predictor of murder. The deck notes that a "more representative" calculation (conditioning on the woman having been murdered) gives about 88%. Lesson: "A numerical argument can look precise while relying on the wrong event."

Extra context beyond the slides: all three are really conditional-probability stories, which is why they are worth remembering when you reach Lecture 4's "check the direction of the bar."

2.2 The basic principle of counting

TWO-STAGE PRODUCT RULE. "If Stage 1 has m outcomes and Stage 2 has n outcomes for every Stage 1 outcome, then the complete two-stage experiment has mn outcomes."

GENERALISED PRINCIPLE. "If r experiments are such that the first may result in any of n_1 possible outcomes; and if, for each of these, there are n_2 possible outcomes of the second; and if, for each of the outcomes of the first two, there are n_3 outcomes of the third; and so on, then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments."

The condition "for every Stage 1 outcome" is the whole point. The number of second-stage choices must not depend on *which* first-stage choice was made, only on *how many* were made. That is what lets you multiply.

Bridge example (slide 29). Roll a die twice: $6 \cdot 6 = 36$ ordered outcomes. Exactly six have sum 7: $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$. If all 36 are equally likely, $\mathbb{P}(\text{sum } 7) = 6/36 = 1/6$. Next lecture: the 36 pairs are a sample space, the six pairs an event.

2.3 The board-work examples, worked

Slide	Problem	Count	Why
30	Two-letter strings, repetition allowed	$26^2 = 676$	26 choices at each position.

31	Two-letter strings, no repetition	$26 \cdot 25 = 650$	Second position has one fewer option whichever letter came first.
33	7-place plate: 3 letters then 4 digits	$26^3 \cdot 10^4 = 175{,}760{,}000$	Seven stages, each independent in count.
34	Same, no repeated letter or digit	$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$	Letters and digits deplete separately.
35	Subsets of an n-student class (including $\emptyset$)	2^n	Each student is either sent or not: n stages with 2 choices. For $n = 2$: $\emptyset, \{1\}, \{2\}, \{1, 2\}$.
39	3 juniors from 5 and 2 seniors from 7	$\binom{5}{3}\binom{7}{2} = 10 \cdot 21 = 210$	Choose each group, then multiply.
39	… but seniors Melissa and John refuse to serve together	$210 - \binom{5}{3} \cdot 1 = 200$	Subtract the committees whose senior pair is exactly {Melissa, John}. Or by cases: $\binom{5}{3}\left[\binom{5}{2} + 2\binom{5}{1}\right] = 10 \cdot 20 = 200$.
46	Bridge deals (4 players, 13 cards each)	$\binom{52}{13,13,13,13} = \dfrac{52!}{(13!)^4} \approx 5.36 \times 10^{28}$	Labelled groups (North, East, South, West) of sizes 13.

2.4 Permutations

PERMUTATIONS. "Consider n distinct objects; how many different ordered arrangements, called permutations, of these objects are there?" Answer $n! = n(n-1)\cdots 2 \cdot 1$, read "$n$ factorial." "Typically, permutations refer to ordered sequences without repetitions." For $n = 2$ objects a, b: ab, ba.

Arranging r of n distinct objects in order (no repetition) is $P(n,r) = n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$. The practice bank uses the notation $P(8,2) = 56$, $P(12,3) = 1320$. Convention: $0! = 1$.

Extra context beyond the slides: your calculator has $n!$ and usually nPr / nCr keys; make sure you know where they are, since you may not use a phone.

2.5 Combinations

COMBINATIONS. "Consider n distinct objects; how many different groups, called combinations, of size r ($1 \leq r \leq n$) can be formed?" Notation $\binom{n}{r}$, read "n choose r." Convention $\binom{n}{0} = 1$. For a, b, c and $r = 2$: $\{a, b\}, \{b, c\}, \{a, c\}$.

Derivation the board work gives: ordered selections of r from n number $n(n-1)\cdots(n-r+1)$, and each unordered group has been counted $r!$ times (once per ordering), so

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}.$$

Useful identities: $\binom{n}{r} = \binom{n}{n-r}$; $\binom{n}{1} = n$; $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ (put $x = y = 1$ in the binomial theorem, and compare with the subsets count).

Watch out: order or not?. HW2 Q7: eight students, a three-person panel with identical roles. A student claims $8 \cdot 7 \cdot 6 = 336$. The error: "the order that the students are chosen in does not matter; this results in identical panels being counted too many times." Each panel was counted $3! = 6$ times, so the answer is $336/6 = \binom{8}{3} = 56$. The reverse error (bank Q9): three *ordered* presentation slots from 8 proposals is $8 \cdot 7 \cdot 6 = 336$, and writing $\binom{8}{3}$ "loses the slot order." Ask: does swapping two chosen items produce a different outcome? Yes → permutation. No → combination.

2.6 The binomial theorem

Binomial theorem.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

The $\binom{n}{r}$ "are also known as binomial coefficients."

Why it is true: expanding $(x+y)^n$ means picking x or y from each of the n factors; the term $x^r y^{n-r}$ arises once for each choice of which r factors supply the x. See §1.3 for the scalar trap.

2.7 Multinomial coefficients

Multinomial coefficient. For integers $n_1, \ldots, n_r \geq 0$ with $n_1 + \cdots + n_r = n$,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

Example: $\binom{4}{1,2,1} = \frac{4!}{1!2!1!} = 12$. With $r = 2$, $\binom{n}{k,\, n-k} = \binom{n}{k}$.

The slides give two interpretations. You must recognise both from wording.

Interpretation 1: labelled groups.
"Number of ways to divide n distinct objects into r distinct groups of sizes $n_1, \ldots, n_r$." Objects a, b, c, d into groups of sizes 1, 1, 2: $\{a\}\{b\}\{c,d\}$, $\{b\}\{a\}\{c,d\}$, ... 12 in all. The groups are *labelled* (Folder A vs Folder B), so $\{a\}\{b\}$ and $\{b\}\{a\}$ are different. Derivation: $\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots$, which telescopes to the formula.

Interpretation 2: arrangements with repeated objects.
"Number of ordered arrangements of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike." A, A, A, B, C, C: $\binom{6}{3,1,2} = 60$ arrangements. Derivation: $6!$ arrangements of six labelled letters, divided by $3!$ for the interchangeable A's and $2!$ for the C's.

Multinomial theorem.

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r):\\ n_1+\cdots+n_r=n}} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}.$$

Worked (HW2 Q10, bank Q8). BANANAS has 7 letters: A×3, N×2, B, S. Arrangements $= \frac{7!}{3!\,2!\,1!\,1!} = \frac{5040}{12} = 420$. BALLOON: L×2, O×2: $\frac{7!}{2!2!} = 1260$.

Worked (HW2 Q12, bank Q7). Nine distinct samples into labelled freezers A (4), B (3), C (2), no order within a freezer: $\binom{9}{4,3,2} = \frac{9!}{4!3!2!} = \frac{362880}{288} = 1260$. Same number as the folders problem $\binom{9}{2,3,4}$, since the denominator is symmetric.

> **WATCH OUT: THE MULTINOMIAL ONLY COUNTS THE ASSIGNMENTS.** Bank Q23: the coefficient of $x^3y^2z^2$ in $(2x - y + 3z)^7$ is *not* $\binom{7}{3,2,2} = 210$. That counts which factors supply each variable; you still multiply by the scalars $2^3(-1)^2 3^2 = 72$, giving 15120.

2.8 The counting decision guide

> **DESCRIBE WHAT ONE OUTCOME LOOKS LIKE, THEN IDENTIFY THE STRUCTURE.**
>
> - sequential choices   multiplication principle;
> - ordered arrangements   permutations;
> - unordered selections   combinations;
> - repeated categories   multinomial coefficients;
> - constrained outcomes   direct counting, cases, or a complement.
>
> "Do not choose a formula from the wording alone."

Diagnose three counting errors (slide 50).

1. "Three different letters are arranged in order from the alphabet: 26^3." Wrong because 26^3 allows repeats. One outcome is an ordered triple of *distinct* letters: $26 \cdot 25 \cdot 24 = 15{,}600$.
2. "Three students are arranged in order from a class of 10: $10!$." Wrong because $10!$ arranges all ten. Only three positions are filled: $10 \cdot 9 \cdot 8 = 720$.
3. "A three-student committee from a class of 10: $10 \cdot 9 \cdot 8$." Wrong because a committee is unordered; each was counted $3!$ times: $\binom{10}{3} = 120$.

Constraint techniques.

- **Complement.** "At least one librarian" among four chosen from 5 librarians + 6 archivists (bank Q5): all panels minus no-librarian panels, $\binom{11}{4} - \binom{6}{4} = 330 - 15 = 315$. HW4 Q8 is the probability version: $1 - \binom{7}{3}/\binom{11}{3} = 1 - 35/165 = 26/33$.
- **Cases.** Pizza with exactly 4 of 8 toppings, never Pepperoni and Pineapple together (bank Q55): neither $\binom{6}{4} = 15$, exactly one $2\binom{6}{3} = 40$; total 55. Party of 5 from 8 friends, two feuding (bank Q53): $\binom{6}{5} + 2\binom{6}{4} = 6 + 30 = 36$.
- **Forced members.** Ana and Bo must both serve on a 5-person committee from 8 (HW1 Q15): choose the other 3 from the remaining 6, $\binom{6}{3} = 20$.
- **Two-stage selection.** Pick a 4-person team from 11, then a lead from the team (bank Q4): $\binom{11}{4} \cdot 4 = 1320$. Contrast with 4 ordered roles: $11 \cdot 10 \cdot 9 \cdot 8 = 7920$.
- **Block trick for adjacency.** Tiles R,R,R,B,B,G,Y with the B's not adjacent (bank Q24): all $7!/(3!2!) = 420$, adjacent (glue BB into one object) $6!/3! = 120$, so 300.
- **Disjoint patterns add.** Lineup FFSSS or FFFSS from 8 faculty and 12 students (bank Q22): $P(8,2)P(12,3) + P(8,3)P(12,2) = 73920 + 44352 = 118272$.
- **Subsets with a forbidden pair.** Subsets of 10 students not containing both Brad and John (bank Q50): $2^{10} - 2^8 = 1024 - 256 = 768$.

2.9 Counting becomes probability

Capstone (slides 51–52). A committee of five is selected uniformly from twelve students: seven juniors, five seniors. $\mathbb{P}(\text{exactly two seniors})$? "Describe the equally likely outcomes, count all outcomes, count the favorable outcomes, and take their ratio."

Sample space and event. Let Ω be the set of all five-student committees and A the event that exactly two seniors are selected.

$$|\Omega| = \binom{12}{5} = 792, \qquad |A| = \binom{5}{2}\binom{7}{3} = 10 \cdot 35 = 350, \qquad \mathbb{P}(A) = \frac{350}{792} = \frac{175}{396} \approx 0.442.$$

Check: $0 \le \mathbb{P}(A) \le 1$.

HW2 Q13. Three-person panel from 5 statistics + 4 CS majors; A = both majors present. $|S| = \binom{9}{3} = 84$. Complement: all-stats $\binom{5}{3} = 10$ plus all-CS $\binom{4}{3} = 4$, so $|A| = 84 - 14 = 70$ and $\mathbb{P}(A) = 70/84 = 0.8333$.

Bank Q10. Four from 5 biology + 4 physics, exactly 2 biology: $\binom{5}{2}\binom{4}{2}/\binom{9}{4} = 60/126 = 10/21$.

Exam tip. The ratio $|A|/|\Omega|$ is only valid when the outcomes you counted are equally likely. Lecture 3 makes that an explicit assumption and bank Q28 (a spinner with unequal sectors) punishes "favourable over total" applied to unequal outcomes. When you count committees, count *unordered* in both numerator and denominator, or *ordered* in both; never mix.

2.10 Extra reading (not examined)

The deck ends with token sequences (V^T strings from a vocabulary of V over T positions, e.g. $5^4 = 625$) and CRISPR guide sequences ($4^{20} \approx 1.1 \times 10^{12}$). Both are the product rule again; the deck's point is that "possible does not mean equally likely." The information sheet excludes Extra Reading, so this is background only.

2.11 Check yourself

1. A submarine sends a three-pulse signal; each pulse is one of five colours, repeats allowed. How many signals? [HW2]

$5^3 = 125$. (The graded HW2 got this wrong; 60 would be "no repeats", 15 would be adding instead of multiplying.)

2. A rover visits four different craters chosen from seven, in order, none twice. How many routes? [HW2]

$P(7, 4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$.

3. Six distinct mechanical birds; three displayed left to right. Which expression: $\binom{6}{3}$, $6 \cdot 5 \cdot 4$, 6^3, or $3!$? [HW2]

$6 \cdot 5 \cdot 4 = 120$. Left-to-right order is recorded, so it is an ordered arrangement of 3 of 6.

4. A playlist has six distinct videos. How many subsets may a student watch, including none? [HW2]

$2^6 = 64$.

5. A committee needs 2 juniors from 6 and 3 seniors from 8, but seniors Priya and Mateo refuse to serve together. How many committees? [HW2]

Juniors: $\binom{6}{2} = 15$. Seniors: $\binom{8}{3} - \binom{6}{1} = 56 - 6 = 50$ (remove trios containing both, which are Priya, Mateo plus one of the other six). Total $15 \cdot 50 = 750$.

6. What is the coefficient of x^3y^5 in $(2x - y)^8$? [HW2]

$\binom{8}{3}2^3(-1)^5 = -448$. Not 56.

7. How many distinct arrangements of the letters of MISSISSIPPI? [L01]

11 letters: I×4, S×4, P×2, M×1. $\frac{11!}{4!4!2!} = \frac{39916800}{1152} = 34650$.

8. Nine students (3 seniors, 3 juniors, 3 sophomores) are split at random into three labelled groups of three. Probability each group has one of each class? [L01 · BANK Q51]

Total labelled splits $\binom{9}{3,3,3} = 1680$. Favourable: assign the three seniors to the three groups one each in $3!$ ways, likewise juniors and sophomores: $(3!)^3 = 216$. $216/1680 \approx 0.1286$.

9. Four dogs and four cats sit at random in eight chairs, four per side. Probability all dogs sit on one side? [L01 · BANK Q48]

$8!$ equally likely seatings. Choose the dogs' side (2), arrange dogs $4!$, cats $4!$: $\frac{2(4!)^2}{8!} = \frac{1152}{40320} = \frac{1}{35} \approx 0.0286$.

10. A student says the number of ways to choose a president, secretary and treasurer from 9 people is $\binom{9}{3} = 84$. Diagnose and correct. [L01]

The roles are distinct, so order matters. Each unordered trio corresponds to $3!$ role assignments: $9 \cdot 8 \cdot 7 = 504$.

11. One subset is chosen uniformly from all subsets of an eight-element set. Probability its size is even? [L01 · BANK Q25]

$1/2$. There are 2^8 subsets; the even-size ones number $\binom{8}{0} + \binom{8}{2} + \cdots + \binom{8}{8} = 2^7$ (add $(1+1)^8$ and $(1-1)^8$).

L02 3. Sample spaces and events

Lecture 2's goal statement: "choose a sample space at the right level of detail, express events as subsets, translate words into event algebra, and check whether a model can answer the question asked." It was taught over two meetings: outcomes, granularity and events first; then event operations, laws, translation and model audits.

3.1 Intuition can mislead: three opening lessons

- **Conjunction fallacy (Linda).** "Linda is a bank teller and active in the feminist movement" cannot be more likely than "Linda is a bank teller."

  CHECK · EVENT INCLUSION. "For any events E and F, $E \cap F \subseteq E$. A plausible narrative cannot reverse this inclusion."

- **Random-looking patterns.** For six fair tosses, $HTHTTH$ and $HHHTTT$ are two individual outcomes of the same sample space; "neither becomes less likely merely because it looks more clustered." The modelling question is whether you are comparing individual sequences or larger events.
- **Coloured-die patterns.** A die with four green and two red faces is rolled 20 times; you win if your pattern appears consecutively: (1) $RGRRR$, (2) $GRGRRR$, (3) $GRRRRR$. Debrief: if $E =$ "$GRGRRR$ appears at a given position" and $F =$ "$RGRRR$ appears one roll later," then $E \subset F$, because the last five symbols of the longer pattern *are* the shorter pattern. So pattern 1 is the best choice and "a pattern may look more representative of randomness without being more likely. Define the outcomes and compare the corresponding events before trusting appearance."

HW3 Q12 is the inclusion lesson in clinic clothing: adding a second condition cannot make $E \cap F$ contain visits outside E; the correct audit is "$E \cap F \subseteq E$ and $E \cap F \subseteq F$; every visit in the intersection satisfies both conditions." **HW3 Q13:** the single sequence $HTHTHT$ is one outcome; "exactly three H's" contains $\binom{6}{3} = 20$ outcomes, so B is 20 times larger than A.

3.2 The five-step modelling template

REAL SYSTEM EXPERIMENT SAMPLE SPACE S EVENTS ASSUMPTIONS. "Real system: What situation matters? Experiment: What is recorded? Sample space S: What outcomes are possible? Events: Which subsets answer our questions? Assumptions: What has been simplified or ruled out?"

GRANULARITY PRINCIPLE. "Record enough detail to distinguish every event the analysis will need."

Two dice (slides 10–11). $S_{\text{pair}} = \{(i, j) : i, j \in \{1, \ldots, 6\}\}$ versus $S_{\text{sum}} = \{2, \ldots, 12\}$. Can each represent (1) the sum is 8, (2) blue exceeds white, (3) the dice match? S_{pair} "can represent sums, comparisons, and matches." S_{sum} "can represent sum questions but loses which die produced which value." The map $(i, j) \mapsto i + j$ "combines many detailed outcomes into one summary. That compression is useful only when no later question needs the discarded information."

Audit a flawed model (slide 25). With $S = \{2, \ldots, 12\}$, the event "first die exceeds the second" is not a subset of S : outcomes $(6, 2)$ and $(2, 6)$ both have sum 8 but disagree about the event. Repair with the smallest natural change: record ordered pairs.

Exam tip: granularity questions are guaranteed. They appear in HW2 Q11 (shuttle count $\{0,1,2\}$ cannot answer "North late and South on time"), HW3 Q14 (parking gates: the ordered pair (n,s) is the only option that supports both the total and the comparison), bank Q11 (sum-only dice), and bank Q20 (route counts $\{0,1,2,3\}$ cannot represent "A late and B on time"; replace by ordered triples $(s_A, s_B, s_C) \in \{O,L\}^3$). The pattern: a count or a sum throws away *which* component did what. Any question about a specific labelled component needs the labelled record.

3.3 Three sizes of sample space

Board work (slide 12): specify S and classify it.

Experiment	Sample space	Type	One assumption
Order in which six named runners finish	All $6! = 720$ orderings	Finite	No ties.
Flip a coin until the first head; record the number of flips	$\{1,2,3,\dots\}$	Countably infinite	Every flip is H or T; flipping continues until a head appears (no upper limit).
Lifetime of a light bulb in hours	$[0,\infty)$	Continuous	Time is measured on a continuum; the bulb eventually fails.

HW3 Q3 and bank Q12 test the "countably infinite" recognition: "number of login attempts until the first success" and "day number of the first alert, with no stated end" are $\{1,2,3,\dots\}$. A lifetime is continuous; three coin tosses or a finishing order is finite.

Stopping rules (bank Q21). A client records S or F per attempt and stops after the first success or the third attempt. The complete records are $\{S, FS, FFS, FFF\}$. Strings like F or FF are unfinished, SSS continues past a success, $FFFF$ exceeds the limit. "These records need not be equally likely."

3.4 Events are subsets

Event. "An event is any subset of S. The event occurs when the observed outcome belongs to that subset." The impossible event is $\emptyset$; the certain event is S.

Two ordered tosses, $S = \{HH, HT, TH, TT\}$: $E = \{\text{both agree}\} = \{HH, TT\}$, $F = \{\text{at least one head}\} = \{HH, HT, TH\}$. HW3 Q2 (sensor RR/RB/BR/BB): "first is red" $\cap$ "reports agree" $= \{RR\}$.

3.5 Union, intersection, complement, difference

Union $E \cup F$.
"outcomes in E, in F, or in both." Means "at least one occurs." "The word *or* is inclusive unless a problem explicitly says otherwise." With $E = \{\text{first toss } H\} = \{HH, HT\}$ and $F = \{\text{tosses differ}\} = \{HT, TH\}$: $E \cup F = \{HH, HT, TH\}$.

Intersection $E \cap F$.
"outcomes in both E and F." Here $E \cap F = \{HT\}$. "E and F" always means intersection.

Disjoint (mutually exclusive).
$E \cap F = \emptyset$. "Disjointness says the two events cannot occur together; it does not say that either event must occur."

Complement E^c.

$E^c = S \setminus E$, the outcomes not in E. For $E = \{\text{first toss } H\}$, $E^c = \{TH, TT\}$. Also $S^c = \emptyset$, $\emptyset^c = S$.

Difference $E \setminus F$.
$E \cap F^c$: E occurs and F does not.

Inclusion $E \subseteq F$.
Every outcome of E is in F, so "E occurs F occurs." $E = F$ exactly when $E \subseteq F$ and $F \subseteq E$.

Watch out: disjoint is not the same as complementary. HW3 Q8: $S = \{1, \ldots, 6\}$, $E = \{1, 2\}$, $F = \{3, 4\}$, $G = \{3, 4, 5, 6\}$. E and F are disjoint but not complements ($E \cup F \neq S$). E and G are disjoint *and* complements. F and G are not disjoint. Bank Q15: north gate N and south gate S cannot both be used (disjoint) but a third gate exists, so "N and S are disjoint but need not be complements." Complements require both $E \cap F = \emptyset$ and $E \cup F = S$.

Countable operations (slide 20). $\bigcup_{n \geq 1} E_n$ = "outcomes in at least one E_n"; $\bigcap_{n \geq 1} E_n$ = "outcomes in every E_n." Translation cue: "Union corresponds to 'at least one index'; intersection corresponds to 'every index.'"

3.6 Three families of laws, and De Morgan

Laws of event algebra.

- Commutative: $E \cup F = F \cup E$, $E \cap F = F \cap E$.
- Associative: $(E \cup F) \cup G = E \cup (F \cup G)$, $(E \cap F) \cap G = E \cap (F \cap G)$.
- Distributive: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ and $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$.

De Morgan's laws.

$$(E \cup F)^c = E^c \cap F^c, \qquad (E \cap F)^c = E^c \cup F^c,$$

and for any finite family $\left(\bigcup E_n\right)^c = \bigcap E_n^c$, $\left(\bigcap E_n\right)^c = \bigcup E_n^c$. "Not 'at least one' means neither"; "not 'both' means at least one failure." Reading rule: "Negate the words first; then switch $\cup \leftrightarrow \cap$ and complement every event."

Superheroes (slide 23). E = Superman saves Metropolis, F = Batman saves Gotham. $E \cup F$ = "at least one city is saved"; $(E \cup F)^c$ = "neither city is saved" = "Superman fails *and* Batman fails" $= E^c \cap F^c$. Likewise $(E \cap F)^c$ = "not both saved" = "at least one of them fails" $= E^c \cup F^c$.

Nested De Morgan (bank Q17). $[A \cup (B \cap C)]^c = A^c \cap (B \cap C)^c = A^c \cap (B^c \cup C^c)$. Apply the law to the outer operation first, then to the inner one.

Three events (bank Q54). $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$. The bank's correct option was written as a union of three copies of that same triple intersection with different bracketing; associativity makes them identical, so their union is still $A^c \cap B^c \cap C^c$. Option "$A^c \cup B^c \cup C^c$" is the complement of the *intersection*, not the union.

3.7 Words symbols

Board work (slide 24), with the standard answers. These are the phrasings that appear on every homework and in the bank.

Words	Symbols
E and F occur, but G does not	$E \cap F \cap G^c$

At least one of E, F, G occurs	$E \cup F \cup G$
All three occur	$E \cap F \cap G$
None occurs (HW3 Q6)	$E^c \cap F^c \cap G^c = (E \cup F \cup G)^c$
Not all three occur	$(E \cap F \cap G)^c = E^c \cup F^c \cup G^c$
Exactly one of E, F (HW3 Q5, bank Q14)	$(E \cap F^c) \cup (E^c \cap F)$
Exactly two of three (bank Q18)	$(E \cap F \cap G^c) \cup (E \cap F^c \cap G) \cup (E^c \cap F \cap G)$
At least two of three (HW3 Q7)	$(E \cap F) \cup (E \cap G) \cup (F \cap G)$
Exactly one of three	$(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G)$
$(E \cup F^c) \cap G$	G occurs, and either E occurs or F does not
$(E \cap F)^c$, two ways	"not both occur" = "at least one fails" = $E^c \cup F^c$

Watch out: "at least two" versus "exactly two". The pairwise-intersection union $(E \cap F) \cup (E \cap G) \cup (F \cap G)$ also contains $E \cap F \cap G$, so it means *at least* two. "Exactly two" needs the third event complemented in each term. Both versions were offered as options in bank Q18 and HW3 Q7.

3.8 Synthesis: coloured balls without replacement

Slide 26: five black and two red balls; draw three without replacement, recording only the ordered colour sequence.

1. **Template.** Experiment: three ordered draws. S = colour strings of length 3 with at most two R's: $\{BBB, BBR, BRB, RBB, BRR, RBR, RRB\}$, seven outcomes. Key assumption: without replacement, so RRR is impossible (only two reds exist). The seven outcomes are *not* equally likely.
2. **Events.** E = second ball black $= \{BBB, BBR, RBB, RBR\}$. F = third black, so F^c = third red $= \{BBR, BRR, RBR\}$. $E \cap F^c = \{BBR, RBR\}$.
3. **Words.** $E \cap F^c$ = "second ball black and third ball red." Check each listed outcome: BBR (second B, third R), RBR (second B, third R).

3.9 Simplify an event expression

Slide 28: simplify $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c)$.

By laws. Distributivity in the form $(X \cup Z) \cap (Y \cup Z) = (X \cap Y) \cup Z$ gives $(A \cup B) \cap (A^c \cup B) = (A \cap A^c) \cup B = \emptyset \cup B = B$. Then $B \cap (A \cup B^c) = (B \cap A) \cup (B \cap B^c) = (A \cap B) \cup \emptyset = A \cap B$.

By membership. An outcome in $A \cap B$ is in all three brackets. An outcome in $A \setminus B$ fails $A^c \cup B$. One in $B \setminus A$ fails $A \cup B^c$. One in neither fails $A \cup B$. So the expression is exactly $A \cap B$.

Bank Q19 is the same technique: $(R \cap S) \cup S^c = (R \cup S^c) \cap (S \cup S^c) = (R \cup S^c) \cap S = R \cup S^c$ (with S here standing for the sample space in the last step). **Bank Q49:** $[(A \cup B) \cap (A \cup C) \cap (B^c \cap C^c)]$: on the region $B^c \cap C^c$, both $A \cup B$ and $A \cup C$ reduce to A, so the event is $A \cap B^c \cap C^c$.

3.10 Verify the model

REPRESENT · TRANSLATE · VERIFY. "Does S contain every recorded outcome and retain the detail we need? Is each event a subset of S, and do its words and symbols agree? Does each identity survive a diagram or membership check?"

Extra reading in this deck (threat circuitry, clusters and John Snow, AI-benchmark coordinates, extreme-heat profiles) is excluded from the exam. Its single transferable lesson is the granularity principle again: a daily-maximum record can represent "max $\geq u$" but not "three consecutive hours $\geq u$."

3.11 Check yourself

1. $S = \{s_1, \ldots, s_6\}$, $E = \{s_1, s_2, s_5\}$, $F = \{s_2, s_3\}$. How many outcomes are in $(E \cup F)^c$? [HW3]

$E \cup F = \{s_1, s_2, s_3, s_5\}$, so the complement is $\{s_4, s_6\}$: 2 outcomes.

2. Booth codes $\Omega = \{h, j, k, m, n, p, q, r\}$, $C = \{h, k, m, q\}$, $D = \{j, k, p, q\}$. Find $(C \setminus D)^c$. [BANK Q16]

$C \setminus D = \{h, m\}$; complement $= \{j, k, n, p, q, r\}$.

3. Write "at most one of A, B, C occurs" in symbols. [L02]

The complement of "at least two": $\left[(A \cap B) \cup (A \cap C) \cup (B \cap C)\right]^c$, equivalently (De Morgan) $(A^c \cup B^c) \cap (A^c \cup C^c) \cap (B^c \cup C^c)$.

4. A model records only the sum of two dice. Which can it answer: (a) sum is prime, (b) doubles, (c) first die is 6? [L02]

Only (a). Doubles and "first die is 6" need the individual dice; e.g. sum 12 is doubles but sum 8 might or might not be.

5. Simplify $(A \cap B) \cup (A \cap B^c)$. [L02 · L03]

$A \cap (B \cup B^c) = A \cap S = A$. This is the decomposition Lecture 3 uses for every identity.

6. Two events are disjoint and $E \cup F = S$. What is F in terms of E? [L02]

$F = E^c$. Both conditions together define complements.

7. Give S for: toss a coin repeatedly until two heads in a row appear; record the number of tosses. Classify it. [L02]

$\{2, 3, 4, \ldots\}$, countably infinite. (You cannot finish in 1 toss.)

8. Is $(E \cup F)^c = E^c \cup F^c$? Give a counterexample if not. [L02]

No. With $S = \{HH, HT, TH, TT\}$, $E = \{HH, HT\}$, $F = \{HT, TH\}$: $(E \cup F)^c = \{TT\}$ but $E^c \cup F^c = \{TH, TT\} \cup \{HH, TT\} = \{HH, TH, TT\}$. The correct law is $E^c \cap F^c = \{TT\}$.

L03 4. Axioms of probability and their consequences

Lecture 3's goals: "audit a probability assignment against the axioms, derive useful probability identities by splitting events into disjoint pieces, and combine exact calculation with independent checks." The opening question: for $S = \{H, T\}$, can we assign $\mathbb{P}(H) = 0.60$ and $\mathbb{P}(T) = 0.60$? "Each number looks plausible by itself. The problem appears only when the assignments are required to describe one coherent model."

4.1 Frequency is evidence, not the definition

TWO DIFFERENT OBJECTS. "The model probability $\mathbb{P}(E)$ is a number assigned before the trials. The empirical proportion $\hat{p}_n(E) = \dfrac{\text{number of occurrences of } E \text{ in } n \text{ trials}}{n}$ is computed from observed data and varies from run to run."

Agreement between the two supports the model; disagreement says to examine the assumptions, the data, or both. Some questions (a future stock price, one person's accident risk) need a model even though the experiment cannot be repeated. HW3 Q15's wrong option "cannot be audited until observed login frequencies are available" is exactly this confusion: validity against the axioms is checked *before* any data.

4.2 The three axioms

THE PROBABILITY AXIOMS. Let S be a sample space and $E, E_1, E_2, \ldots$ events.

1. **Nonnegativity:** $\mathbb{P}(E) \geq 0$ for every event E.
2. **Normalisation:** $\mathbb{P}(S) = 1$.
3. **Countable additivity:** if $E_1, E_2, \ldots$ are pairwise disjoint, then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$.

"Coherence: the conditions constrain all event probabilities simultaneously; an assignment is not valid merely because each number lies between zero and one."

Kolmogorov's 1933 monograph gave these axioms. A **probability space** is $(S, \text{events}, \mathbb{P})$: a sample space, an allowed collection of events, and an assignment satisfying the axioms. "For finite models in this course, every subset of S may be treated as an event." Three roles: "Assumptions choose S, the events, and the point masses Axioms constrain the assignment Theorems derive probabilities of new events."

4.3 First consequences

CONSEQUENCE 1. $\mathbb{P}(\emptyset) = 0$. Proof: $S = S \,\dot{\cup}\, \emptyset$, so additivity gives $\mathbb{P}(S) = \mathbb{P}(S) + \mathbb{P}(\emptyset)$.

CONSEQUENCE 2 (FINITE ADDITIVITY). If $E_1, \ldots, E_n$ are pairwise disjoint then $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$. "This finite rule follows by adjoining empty events to the infinite sequence."

The symbol $\dot{\cup}$ means "union of disjoint sets" and is the instructor's cue that additivity applies.

4.4 Auditing an assignment

Coin audit (slide 8). $S = \{H, T\}$, $0 < p < 1$.

1. $\mathbb{P}(H) = p,\ \mathbb{P}(T) = 1 - p$: nonnegative, sums to 1. Valid.
2. $\mathbb{P}(H) = p,\ \mathbb{P}(T) = 1 - p^2$: nonnegative, but the sum is $1 + p - p^2 > 1$ for $0 < p < 1$. Fails normalisation (via finite additivity). Invalid.
3. $\mathbb{P}(H) = 1.10,\ \mathbb{P}(T) = -0.10$: fails nonnegativity at $\mathbb{P}(T)$. Invalid. (The sum happens to be 1; that does not rescue it.)

Audit sequence. "Check nonnegativity first. Then add the singleton masses and compare their total with normalization. Stop at the first failed condition and justify that diagnosis."

HW3 Q15. $\mathbb{P}(N) = 0.52, \mathbb{P}(R) = 0.31, \mathbb{P}(L) = 0.22$: "invalid because the three disjoint singleton probabilities sum to 1.05, contradicting normalization and finite additivity." **Bank Q26** is the same with four states summing to 1.05. The distractor "a valid model must assign the same probability to all outcomes" is false: equal likelihood is an assumption, not an axiom.

4.5 Finite models are built from point masses

Point-mass model. Let $S = \{x_1, \ldots, x_N\}$. Choose $p_i \geq 0$ with $\sum_{i=1}^{N} p_i = 1$ and set

$$\mathbb{P}(E) = \sum_{i:\, x_i \in E} p_i.$$

"The point masses p_i represent the experiment; the axioms do not choose them." Once fixed, finite additivity determines every event probability. Fair die: $p_1 = \cdots = p_6 = 1/6$.

Unfair die (slide 10). $\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(3) = \frac{1}{4}$, $\mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6) = \frac{1}{12}$. (1) Valid: all nonnegative and $3 \cdot \frac{1}{4} + 3 \cdot \frac{1}{12} = \frac{3}{4} + \frac{1}{4} = 1$. (2) $\mathbb{P}(\text{even}) = \mathbb{P}(2) + \mathbb{P}(4) + \mathbb{P}(6) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$. (3) Quick bound: it must lie between $\mathbb{P}(2) = \frac{1}{4}$ and $1 - \mathbb{P}(1) - \mathbb{P}(3) = \frac{1}{2}$.

HW3 Q16. $\mathbb{P}(G) = c, \mathbb{P}(Y) = 2c, \mathbb{P}(O) = 3c, \mathbb{P}(R) = 4c$. Normalisation: $10c = 1$, $c = 0.1$. $\mathbb{P}(\{Y, R\}) = 0.2 + 0.4 = 0.6$. **Bank Q27:** $\mathbb{P}(\{D, O\}) = 0.28 + 0.25 = 0.53$, because singletons are disjoint.

The simulation slide's workflow: "Predict using your board calculation → Simulate repeated rolls → Compare the empirical proportion with your exact value → Explain ordinary variation or a model mismatch." A finite empirical proportion need not equal the exact value.

4.6 Equally likely outcomes

Equally likely model. If $S = \{x_1, \ldots, x_N\}$ and the model assigns $\mathbb{P}(x_i) = 1/N$ to every outcome, then

$$\mathbb{P}(E) = \frac{|E|}{|S|}.$$

"The ratio 'favorable outcomes divided by total outcomes' is valid only after the outcomes in S have been shown or assumed to be equally likely." "Combinatorics supplies $|E|$ and $|S|$; the probability model justifies taking their ratio."

Watch out: named categories are not automatically equally likely. Bank Q28: a wheel with sectors of $60°, 60°, 90°, 150°$, the first two blue. $\mathbb{P}(\text{blue}) \neq 2/4$. Equal likelihood applies to the *angle*, so $\mathbb{P}(\text{blue}) = 120/360 = 1/3$. HW4 Q2: ten labelled alert messages (4 delay, 3 detour, 2 elevator, 1 crowding). Model I (uniform over messages) gives $\mathbb{P}(\text{delay}) = 0.40$; Model II (uniform over the four categories) gives 0.25. Both are valid models; they answer different questions. Read which objects are uniform.

4.7 Sampling two people: three models that agree (or nearly)

900 adults: 600 Democrats, 300 Republicans. Two names selected uniformly. Event: different party labels.

- **Unordered, without replacement (slide 13).** S = all pairs of distinct adults, $|S| = \binom{900}{2} = 404{,}550$, each equally likely. Mixed pairs: $600 \cdot 300 = 180{,}000$. $\mathbb{P} = 180000/404550 \approx 0.4449$.
- **Ordered, without replacement (slide 14).** $|S| = 900 \cdot 899$, each ordered pair probability $1/(900 \cdot 899)$. Mixed: $600 \cdot 300 + 300 \cdot 600 = 360{,}000$. $\mathbb{P} = 360000/809100 \approx 0.4449$. It must agree because each unordered pair corresponds to exactly two ordered pairs, in both numerator and denominator.
- **With replacement (slide 15).** $S_{\text{with}} = \{(u, v)\}$, $\mathbb{P}((u, v)) = 1/900^2$; repeated names now possible. Mixed: $360000/810000 = 0.4444$. Slightly smaller: the first draw no longer removes a person of its own party.

HW4 Q4 (route cards 1–8, two draws). Model W (with replacement): $|S| = 64$, $\mathbb{P}(\text{match}) = 8/64 = 1/8$. Model N (without): $|S| = 56$, $\mathbb{P}(\text{match}) = 0$. **HW4 Q3:** 5 high-capacity of 12 packs, exactly one high in two draws: without replacement $\binom{5}{1}\binom{7}{1}/\binom{12}{2} = 35/66 = 0.5303$; with replacement $2 \cdot \frac{5}{12} \cdot \frac{7}{12} = 0.4861$; difference 0.0442. **Bank Q29:** 3 blue, 2 gold; one of each: without $\frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}$; with $2 \cdot \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{25}$.

Exam tip. "Without replacement" pushes mixed-colour (or "exactly one") probabilities *up* and matching probabilities *down* relative to "with replacement," because the first draw depletes its own category. Use that direction as a sanity check on multiple-choice pairs.

4.8 One decomposition drives every identity

Reusable method. For any events A, B:

$$A = (A \cap B) \,\dot{\cup}\, (A \cap B^c), \qquad \text{so} \qquad \mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$

"Partition an event into disjoint pieces, apply additivity, and then simplify."

Complement rule. $S = E \,\dot{\cup}\, E^c$, so $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.

Monotonicity. If $E \subseteq F$ then $F = E \,\dot{\cup}\, (F \cap E^c)$, so $\mathbb{P}(F) = \mathbb{P}(E) + \mathbb{P}(F \cap E^c) \geq \mathbb{P}(E)$.

Bank Q30. $A \subseteq B$, $\mathbb{P}(A) = 0.57$, $\mathbb{P}(B^c) = 0.18$. Then $\mathbb{P}(B) = 0.82$ and $\mathbb{P}(B \cap A^c) = 0.82 - 0.57 = 0.25$.

4.9 Two-event inclusion–exclusion

Union of two events. $E \cup F = E \,\dot{\cup}\, (F \cap E^c)$, and additivity gives

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

"The overlap is subtracted once because it was counted in both $\mathbb{P}(E)$ and $\mathbb{P}(F)$."

Campus services (slide 20). 80% used Recreation or a Union program; 60% Recreation, 25% Union. Overlap $= 0.60 + 0.25 - 0.80 = 0.05$. Upper bound before calculating: overlap $\leq \min(0.60, 0.25) = 0.25$. Union check: 0.80 lies between $\max(0.60, 0.25) = 0.60$ and $\min(1, 0.85) = 0.85$.

Audit checks.

$$0 \leq \mathbb{P}(R \cap U) \leq \min\{\mathbb{P}(R), \mathbb{P}(U)\}, \qquad \max\{\mathbb{P}(R), \mathbb{P}(U)\} \leq \mathbb{P}(R \cup U) \leq \min\{1, \mathbb{P}(R) + \mathbb{P}(U)\}.$$

Bank Q32. $\mathbb{P}(R) = 0.72, \mathbb{P}(U) = 0.41$: $\mathbb{P}(R \cup U) \in [0.72, 1]$, so among the offered values only 0.88 is feasible (overlap 0.25).

"Exactly one" of two events. The union is the disjoint union of the exactly-one region and the overlap, so

$$\mathbb{P}(\text{exactly one}) = \mathbb{P}(E \cup F) - \mathbb{P}(E \cap F) = \mathbb{P}(E) + \mathbb{P}(F) - 2\mathbb{P}(E \cap F).$$

HW4 Q5: $\mathbb{P}(B) = 0.62, \mathbb{P}(S) = 0.47, \mathbb{P}(B \cup S) = 0.81 \to$ overlap 0.28, exactly one $0.81 - 0.28 = 0.53$. Bank Q31: $0.64 + 0.46 - 0.82 = 0.28$, exactly one 0.54.

HW3 Q9 (dice-event union), corrected. Blue and white dice, $|S| = 36$. A = blue odd: $|A| = 18$. B = sum ≥ 9: sums 9, 10, 11, 12 have 4, 3, 2, 1 outcomes, $|B| = 10$. $A \cap B$: blue 3 with white 6; blue 5 with white 4, 5, 6: $|A \cap B| = 4$. $|A \cup B| = 18 + 10 - 4 = 24$, $\mathbb{P} = 24/36 = 0.6667$. The graded copy entered 0.5556 ($= 20/36$) and lost the point; recount $|A \cap B|$ by listing.

4.10 Three-event inclusion–exclusion

Union of three events.

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G) - \mathbb{P}(E \cap F) - \mathbb{P}(E \cap G) - \mathbb{P}(F \cap G) + \mathbb{P}(E \cap F \cap G).$$

Audit by membership pattern (slide 23). An outcome in exactly one event is counted 1 time. In exactly two: $2 - 1 = 1$. In all three: $3 - 3 + 1 = 1$. In none: 0. So every outcome of the union is counted exactly once.

Course enrollment (slides 24–25). 100 students; $|P| = 65, |L| = 45, |C| = 40$, $|P \cap L| = 30, |P \cap C| = 25, |L \cap C| = 20$, $|P \cap L \cap C| = 10$.

- Union: $65 + 45 + 40 - 30 - 25 - 20 + 10 = 85$. None: $1 - 0.85 = 0.15$.
- Exactly one: P only $= 65 - 30 - 25 + 10 = 20$ (the triple was subtracted twice, add it back once); L only $= 45 - 30 - 20 + 10 = 5$; C only $= 40 - 25 - 20 + 10 = 5$. Total 30, probability 0.30.

"Only" and "exactly two" counts.

$$|A \text{ only}| = |A| - |A\cap B| - |A\cap C| + |A\cap B\cap C|, \qquad |A\cap B \text{ only}| = |A\cap B| - |A\cap B\cap C|.$$

Exactly two $=$ sum of the three pair-only counts. Exactly one $=$ sum of the three "only" counts. None $= |S| - |A\cup B\cup C|$.

HW4 Q6: none of three flags $= 1-(0.46+0.39+0.31-0.18-0.14-0.12+0.05) = 1-0.77 = 0.23$. **HW4 Q7:** pair-only counts $55-25=30$, $45-25=20$, $40-25=15$; exactly two $=65/240=0.2708$. **HW4 Q9:** A only $=95-40-30+15=40$, B only $=80-40-25+15=30$, C only $=70-30-25+15=30$; exactly one $=100/200=0.500$. **Bank Q33:** none $=1-0.78=0.22$. **Bank Q46 (Olympics):** at least one $=0.28+0.29+0.19-0.14-0.12-0.10+0.08=0.48$, none $=0.52\in(0.5,0.6]$.

Exam tip. Draw the three-circle Venn diagram and fill from the centre outward: triple first, then each pair minus the triple, then each single minus its two pair-only regions minus the triple. Every "exactly k" question is then reading off regions. This is faster and safer than the formula under time pressure.

4.11 Choosing groups: unordered versus sequential

Three-student group (slide 29). Six first-years, four sophomores; group of three uniform over $\binom{10}{3}=120$. One first-year and two sophomores: $\binom{6}{1}\binom{4}{2}=6\cdot 6=36$; $\mathbb{P}=36/120=0.3$.

Sequential check (slide 30). Ordered draws: $10\cdot 9\cdot 8=720$ sequences. Favourable: choose which draw is the first-year (3 ways), fill it (6), fill the other two with sophomores ($4\cdot 3$): $3\cdot 6\cdot 12=216$. $216/720=0.3$. Same answer, as it must be.

Tablet giveaway (slide 31). 120 people, six names drawn without replacement. $\mathbb{P}(\text{you win})=\binom{119}{5}/\binom{120}{6}=6/120=1/20$. General: $\binom{n-1}{k-1}/\binom{n}{k}=k/n$. HW4 Q10 (Maya, 4 of 18): by symmetry $4/18$; by counting $\binom{17}{3}/\binom{18}{4}=2/9$. The option "$1-(17/18)^4$" is wrong because draws are without replacement, not repeated independent selections.

Deer (bank Q47). 300 deer, 100 tracked; sample 3 without replacement; $\mathbb{P}(\leq 1 \text{ tracked}) = \dfrac{\binom{200}{3}+\binom{100}{1}\binom{200}{2}}{\binom{300}{3}}$. "At most one" = zero *or* exactly one: two disjoint counts added.

4.12 Birthday collisions

Assume birthdays independent and uniform over 365 days (no February 29). For n students, $\mathbb{P}(\text{at least two share})$? "Count the complement event in which all birthdays are different."

Birthday formula.

$$\mathbb{P}(\text{no shared birthday}) = \frac{365\cdot 364\cdots(365-n+1)}{365^n} = \prod_{j=0}^{n-1}\Bigl(1-\frac{j}{365}\Bigr), \qquad \mathbb{P}(\text{some shared}) = 1-\text{that}.$$

Values: $n=22$: 0.476; $n=23$: 0.507; $n=100$: ≈ 0.9999997. The probability first exceeds one half at $n=23$.

Denominator: 365^n ordered assignments, equally likely. Numerator: ordered assignments with all distinct days, $P(365,n)$. **HW4 Q11:** the product $365\cdot 364\cdots 350/365^{16}$ is $\mathbb{P}(\text{all 16 different})$, so the shared-birthday probability is $1-$ that. The distractor "denominator should be $\binom{365}{16}$" mixes ordered numerator with unordered denominator. **Bank Q34:** 12 files, 20 suffixes: $\mathbb{P}(\text{collision}) = 1 - \dfrac{20 \cdot 19 \cdots 9}{20^{12}}$. **Bank Q52:** 5 people, $\mathbb{P}(\text{all distinct}) = \dfrac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365^5} \approx 0.9729$.

Bulgarian lottery (slide 34). The same six numbers came up on September 6 and 10, 2009. A *specified* later draw matches a fixed draw with probability $1/\binom{49}{6} = 1/13{,}983{,}816$; but across many drawings there are many pairs that could match, and about 4,404 drawings make a repeat more likely than not. "Repeat events need the right comparison set." **V-1 impacts:** visible clustering "does not prove targeting"; random spatial models also produce clusters.

4.13 Poker hands

Full house. Five cards dealt uniformly; three of one rank and two of another.

$$\mathbb{P}(\text{full house}) = \frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2{,}598{,}960} \approx 0.001441.$$

"Choose the triple's rank and suits, then the pair's different rank and suits. The denominator assumes all five-card hands are equally likely."

Two pairs (bank Q35). $\dfrac{\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4}{\binom{52}{5}} = \dfrac{123552}{2598960} \approx 0.0475$. The two pair ranks are chosen *together* with $\binom{13}{2}$ (unordered), the fifth card from the 11 remaining ranks.

Watch out. For a full house the ranks are $13 \cdot 12$, ordered, because the triple rank and the pair rank play different roles. For two pairs the ranks are $\binom{13}{2}$, unordered, because the two pairs play the same role. Ask: "does swapping the two roles give a different hand?"

4.14 Aside: Maxwell–Boltzmann and Bose–Einstein

Marked "context, not syllabus" on the slides. r labelled particles in n cells: n^r ordered configurations (Maxwell–Boltzmann). Recording only occupation numbers $(r_1, \ldots, r_n)$ with $\sum r_i = r$: stars and bars gives $\binom{r+n-1}{n-1}$ vectors (Bose–Einstein). Which outcomes are equally likely is a modelling assumption, not an axiom. The extra reading (hash collisions, XKCD multiple testing, union bound $\mathbb{P}(\bigcup F_i) \le \sum \mathbb{P}(F_i)$, Bonferroni α/k) is excluded.

4.15 Check yourself

1. $S = \{a, b, c\}$ with $\mathbb{P}(a) = 0.5, \mathbb{P}(b) = 0.3, \mathbb{P}(c) = 0.3$. Valid? Name the first failing condition. [L03]

Nonnegativity holds; the masses sum to 1.1, so finite additivity gives $\mathbb{P}(S) = 1.1 \neq 1$. Fails normalisation. Invalid.

2. $\mathbb{P}(E) = 0.55$, $\mathbb{P}(F) = 0.35$, $\mathbb{P}(E \cap F) = 0.20$. Find $\mathbb{P}(E \cup F)$, $\mathbb{P}(E^c \cap F^c)$, $\mathbb{P}(E \cap F^c)$, and $\mathbb{P}(\text{exactly one})$. [L03]

$\mathbb{P}(E \cup F) = 0.70$; neither $= 0.30$; $\mathbb{P}(E \cap F^c) = 0.55 - 0.20 = 0.35$; exactly one $= 0.70 - 0.20 = 0.50$.

3. $\mathbb{P}(R) = 0.6$, $\mathbb{P}(U) = 0.7$. Which are possible values of $\mathbb{P}(R \cap U)$: 0.25, 0.3, 0.65? [L03]

Upper bound $\min(0.6, 0.7) = 0.6$, so 0.65 is impossible. Lower bound: $\mathbb{P}(R \cup U) \leq 1$ forces $\mathbb{P}(R \cap U) \geq 0.6 + 0.7 - 1 = 0.3$, so 0.25 is impossible too. Only 0.3 works.

4. Fair die rolled twice. Probability the two rolls differ? Probability the max is 4? [L03]

Differ: $1 - 6/36 = 5/6$. Max exactly 4: outcomes with both ≤ 4 minus both ≤ 3: $16 - 9 = 7$, so $7/36$.

5. Among 200 requests, $|A| = 95, |B| = 80, |C| = 70, |A \cap B| = 40, |A \cap C| = 30, |B \cap C| = 25, |A \cap B \cap C| = 15$. Find $\mathbb{P}(\text{exactly two})$ and $\mathbb{P}(\text{none})$. [HW4]

Pair-only: $25, 15, 10 \to$ exactly two $= 50/200 = 0.25$. Union $= 95 + 80 + 70 - 40 - 30 - 25 + 15 = 165 \to$ none $= 35/200 = 0.175$. (Check: exactly one 100 + exactly two 50 + all three 15 = 165 .)

6. A five-card hand: probability of four of a kind (four cards of one rank plus one other card)? [L03]

$13 \cdot \binom{4}{4} \cdot 48 / \binom{52}{5} = 624/2598960 \approx 0.00024$.

7. Eight people; probability at least two share a birth *month* (12 months equally likely)? [L03]

$$1 - \frac{12 \cdot 11 \cdot 10 \cdots 5}{12^8} = 1 - \frac{19958400}{429981696} \approx 0.954.$$

8. Nine samples: four red, three blue, two green; three chosen uniformly. $\mathbb{P}(\text{at least two colours})$? [HW3]

Complement: single colour $= \binom{4}{3} + \binom{3}{3} + 0 = 5$ of $\binom{9}{3} = 84$. $\mathbb{P} = 79/84 \approx 0.9405$.

9. A student claims $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ always. When is that true, and what is the general formula? [L03]

Only when $E \cap F = \emptyset$ (or more generally $\mathbb{P}(E \cap F) = 0$). In general $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

L04 5. Conditional probability

Lecture 4's goals: "identify the observed information, restrict the sample space, compute a conditional probability, use multiplication along a sequence, and read a probability tree." The two question columns on the slide: *Interpret* (What information was observed? Which outcomes remain possible?) and *Calculate and check* (What is the new denominator? Did we reverse the conditional?).

5.1 Update before using a formula

Three red balls (slide 3). A friend uses one of two boxes: M_1: 2 red, 8 white; M_2: 8 red, 2 white. You draw three without replacement and all three are red. Which model remains possible? M_1 has only two red balls, so three reds is impossible under M_1. Under the stated assumptions you can be *certain* it is M_2. The lesson (Bayes and Price, Laplace): "Observed information changes the probabilities that matter." No formula was needed, only restriction of the possible models.

5.2 Restrict the table when information arrives

The modelled class of 100 students:

	In state	Out of state	Total
STAN	52	18	70
Non-STAN	15	15	30
Total	67	33	100

Before conditioning, $\mathbb{P}(\text{In state}) = 67/100$ and $\mathbb{P}(\text{STAN}) = 70/100$; "every one of the 100 students is still in the reference population."

1. Given STAN, $\mathbb{P}(\text{In state} \mid \text{STAN})$: target = in state, observed = STAN, remaining students = the 70 in the STAN row, new denominator 70. Answer $52/70 \approx 0.743$.
2. Given non-STAN: remaining = 30, answer $15/30 = 0.5$.

> TRACK EXPLICITLY. "the target, the observed information, the remaining students, and the new denominator."

5.3 The definition: intersect, then renormalise

> CONDITIONAL PROBABILITY. For events E and F with $\mathbb{P}(F) > 0$,
>
> $$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$
>
> *Restriction:* "the phrase 'given F' makes F the new universe; outcomes outside F are discarded." *Renormalisation:* "the numerator retains the outcomes in both E and F; division by $\mathbb{P}(F)$ makes the new total one." The ratio is undefined when $\mathbb{P}(F) = 0$.

In an equally likely model this is just $\mathbb{P}(E \mid F) = |E \cap F|/|F|$: count the survivors in F, then count how many of them are also in E.

WATCH OUT: THE DIRECTION OF THE BAR. Slide 9: a student writes $\mathbb{P}(\text{In state} \mid \text{STAN}) = \mathbb{P}(\text{STAN} \mid \text{In state})$, "arguing that both questions involve the same 52 students in the intersection." The numerators agree, but the denominators differ: $52/70 = 0.743$ versus $52/67 = 0.776$. HW4 Q12: $\mathbb{P}(C \mid E) = 48/80 = 0.60$ but $\mathbb{P}(E \mid C) = 48/75 = 0.64$. Bank Q38: $\mathbb{P}(E \mid L) = 0.10/0.40 = 0.25$ and $\mathbb{P}(L \mid E) = 0.10/0.25 = 0.40$. Always divide by the probability of the event *after* the bar.

5.4 List the restricted universe

At least one girl (slide 10). Two children, ordered older/younger, $S = \{BB, BG, GB, GG\}$ equally likely. A = at least one girl $= \{BG, GB, GG\}$, B = both girls $= \{GG\}$. $\mathbb{P}(B \mid A) = \mathbb{P}(B \cap A)/\mathbb{P}(A) = (1/4)/(3/4) = 1/3$. Three outcomes survive conditioning; one of them is GG. (Contrast: given the *older* child is a girl, survivors are $\{GB, GG\}$ and the answer is $1/2$.)

Two hearts given at least one heart (slide 11). Two cards without replacement. B = both hearts, A = at least one heart, so $B \subseteq A$ and $B \cap A = B$. Numerator $|B| = \binom{13}{2} = 78$. Restricted denominator by complement: $|A| = \binom{52}{2} - \binom{39}{2} = 1326 - 741 = 585$. $\mathbb{P}(B \mid A) = 78/585 = 2/15 \approx 0.133$.

Bank Q39. Two spinners $\{1, 2, 3\}$, nine equally likely pairs. Given at least one 3, survivors are $(3,1), (3,2), (3,3), (1,3), (2,3)$: five. Sum 4: $(3,1), (1,3)$. $\mathbb{P} = 2/5$. **Bank Q43:** committee of 3 from 4 stats + 3 OR; given at least one OR, exactly one OR: $\binom{3}{1}\binom{4}{2}/\left[\binom{7}{3} - \binom{4}{3}\right] = 18/31$.

EXAM TIP. When the conditioning event is "at least one ...", count its size by complement ($|S| - |\text{none}|$). When the target is a subset of the conditioning event (both hearts   at least one heart), the numerator is just the target's count.

5.5 The multiplication rule

MULTIPLICATION RULE.

$$\mathbb{P}(E \cap F) = \mathbb{P}(F)\,\mathbb{P}(E \mid F) \quad (\mathbb{P}(F) > 0), \qquad \mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F \mid E) \quad (\mathbb{P}(E) > 0).$$

"Same joint event, different order: begin with one event, then multiply by the probability of the other event given what is already known." Translation: "the phrase 'E and F' describes the intersection $E \cap F$."

Actuarial exams (slide 15). Pass June with 0.9; given a June pass, pass July with 0.8. "Passes both" $= J \cap L$, $\mathbb{P}(J \cap L) = \mathbb{P}(J)\mathbb{P}(L \mid J) = 0.9 \cdot 0.8 = 0.72$. The conditional factor is the July pass. **Bank Q56:** John attends 0.65, Joe attends given John 0.8: both $= 0.52$. **HW4 Q14:** $\mathbb{P}(A \cap D) = 0.37 \cdot 0.68 = 0.2516$. **Bank Q37:** $0.65 \cdot 0.40 = 0.26$.

Reversing with the definition (HW4 Q15, HW4 Q13, bank Q45). You are given $\mathbb{P}(E)$, $\mathbb{P}(B \mid E)$ and $\mathbb{P}(B)$ and asked for $\mathbb{P}(E \mid B)$. Two steps: multiplication rule for the intersection, then the definition with B as the new universe.

- HW4 Q15: $\mathbb{P}(E \cap B) = \frac{29}{50} \cdot \frac{13}{20} = \frac{377}{1000}$; $\mathbb{P}(E \mid B) = \frac{377/1000}{49/100} = \frac{377}{490} \approx 0.7694$. (The graded copy entered the decimal; the question asked for the exact fraction $377/490$, already in lowest terms.)
- HW4 Q13: $\mathbb{P}(R \cap S) = 0.82 \cdot 0.70 = 0.574$; $\mathbb{P}(R \mid S) = 0.574/0.61 = 0.9410$.
- Bank Q45: $\mathbb{P}(A \cap B) = 0.24 \cdot 0.75 = 0.18$; $\mathbb{P}(A \mid B) = 0.18/0.45 = 0.40$; $\mathbb{P}(A^c \mid B) = 1 - 0.40 = 0.60$. The complement rule works inside the conditioned universe.

Extra context beyond the slides: this two-step manoeuvre is Bayes' rule in disguise, but the exam excludes Bayes *as a named formula*. Using multiplication then the definition is Lecture 4 material and is exactly what HW4 Q15 asked for.

5.6 The chain rule

CHAIN RULE.

$$\mathbb{P}(E_1 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\,\mathbb{P}(E_2 \mid E_1)\,\mathbb{P}(E_3 \mid E_1 \cap E_2) \cdots \mathbb{P}(E_n \mid E_1 \cap \cdots \cap E_{n-1}),$$

provided the conditional probabilities are defined. "Each factor is the probability of the next step given every earlier step on that path." Common error: "Without replacement, the available population changes after each draw, so later factors usually change."

With versus without replacement (slide 14). 50 students, 20 prefer probability. Both selected prefer probability: without replacement $\frac{20}{50} \cdot \frac{19}{49} = \frac{380}{2450} \approx 0.1551$; with replacement $\frac{20}{50} \cdot \frac{20}{50} = 0.16$. The second factor is what changes.

Bank Q42. 5 blue, 3 red, 2 gold; blue then red then gold without replacement: $\frac{5}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} = \frac{30}{720} = \frac{1}{24}$. **Bank Q40:** 4 green, 2 gold; $\mathbb{P}(\text{2nd gold} \mid \text{1st gold})$ is $1/5$ without replacement and $2/6 = 1/3$ with. **Bank Q44:** the student wrote $\mathbb{P}(R_1 \cap B_2) = \frac{3}{5} \cdot \frac{2}{5}$; the second factor must be $\mathbb{P}(B_2 \mid R_1) = \frac{2}{4}$, giving $\frac{3}{10}$.

EXAM TIP. Chain-rule answers for ordered colour sequences equal the unordered hypergeometric count divided by the number of orderings. Check: $\mathbb{P}(\text{one gold, one green in two draws})$ from bank Q40's bag is $\frac{2}{6} \cdot \frac{4}{5} + \frac{4}{6} \cdot \frac{2}{5} = \frac{16}{30} = \binom{2}{1}\binom{4}{1}/\binom{6}{2}$. If your tree answer and your counting answer disagree, one of them is wrong.

5.7 Probability trees: multiply along, add across

The response problem (slides 16–18). Shankar picks one of two STOR 435 sections uniformly, then emails a uniformly chosen student from that section. Section 01 is 30% actuarial, Section 02 is 50%. An actuarial student replies with probability 0.30 in Section 01 (midterm week) and 0.80 in Section 02. A non-actuarial student never replies.

```
Start
├── Section 01 (0.50)
│   ├── Actuarial (0.30) ── Reply (0.30)   → 0.50 × 0.30 × 0.30 = 0.045
│   └── Non-actuarial (0.70) ── Reply (0)
└── Section 02 (0.50)
    ├── Actuarial (0.50) ── Reply (0.80)   → 0.50 × 0.50 × 0.80 = 0.200
    └── Non-actuarial (0.50) ── Reply (0)
```

TREE GRAMMAR. "Edges carry conditional probabilities. Multiply along one path; add only across disjoint terminal paths."

$\mathbb{P}(\text{Section 01 and reply}) = 0.045$. Overall $\mathbb{P}(\text{reply}) = 0.045 + 0.200 = 0.245$. The two routes are disjoint because a student is selected from only one section. (Lecture 5 names this "total probability"; on this exam you just need the tree.)

"Exactly one" on a tree (bank Q41, Q57, HW-style). Reading done with 0.70; office hours given reading 0.90, given no reading 0.40. Exactly one of the two events: reading-and-no-OH $0.70 \cdot 0.10 = 0.07$ plus no-reading-and-OH $0.30 \cdot 0.40 = 0.12$: total 0.19. Bank Q57: John 0.65, Joe given John 0.8, Joe given no John 0.45: exactly one $= 0.65 \cdot 0.2 + 0.35 \cdot 0.45 = 0.13 + 0.1575 = 0.2875$.

WATCH OUT. Second-level branch probabilities are conditional on the first level; they need not be the same on each branch (0.30 vs 0.80 above). And the probabilities leaving any one node must sum to 1 (0.30 + 0.70 for Section 01), which is a quick audit of a tree you have drawn.

5.8 Conditioning checklist

BEFORE CALCULATING. 1. Name the target event. 2. Name the observed event. 3. Restrict the universe. 4. Identify the new denominator.

AFTER CALCULATING. 1. Check the answer lies in $[0, 1]$. 2. Check the direction of the bar. 3. Compare with a table, count, or tree.

The extra reading (the brain as a betting engine, $\mathbb{P}(\text{cause} \mid \text{cue}) \propto \mathbb{P}(\text{cue} \mid \text{cause})\mathbb{P}(\text{cause})$, fear and high stakes, prediction error) is excluded from the exam. Lecture 5 (total probability, Bayes, independence) is excluded too.

5.9 Check yourself

1. From the 100-student table, find $\mathbb{P}(\text{STAN} \mid \text{Out of state})$ and $\mathbb{P}(\text{Out of state} \mid \text{STAN})$. [L04]

$18/33 \approx 0.545$ and $18/70 \approx 0.257$.

2. Mentoring data: first-years 18 attended, 12 did not; others 22 attended, 48 did not. $\mathbb{P}(\text{first-year} \mid \text{attended})$? [BANK Q36]

Attendees $= 40$; $18/40 = 0.45$.

3. Roll two fair dice. Given the sum is 7, probability one die shows 3? Given one die shows 3, probability the sum is 7? [L04]

Sum 7 has 6 outcomes; two contain a 3 ($(3, 4), (4, 3)$): $2/6 = 1/3$. At least one 3 has 11 outcomes; two of them sum to 7: $2/11$.

4. $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B \mid A) = 0.25$, $\mathbb{P}(B) = 0.2$. Find $\mathbb{P}(A \cap B)$, $\mathbb{P}(A \mid B)$, $\mathbb{P}(A^c \mid B)$. [L04]

0.10; $0.10/0.2 = 0.5$; 0.5.

5. Three cards dealt without replacement. $\mathbb{P}(\text{all three aces})$? [L04]

$\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{24}{132600} \approx 0.000181$. Same as $\binom{4}{3}/\binom{52}{3} = 4/22100$.

6. Factory A makes 60% of parts with 2% defective; factory B makes 40% with 5% defective. Draw the tree and find $\mathbb{P}(\text{defective})$ and $\mathbb{P}(\text{from A and defective})$. [L04]

$\mathbb{P}(A \cap D) = 0.6 \cdot 0.02 = 0.012$; $\mathbb{P}(B \cap D) = 0.4 \cdot 0.05 = 0.020$; $\mathbb{P}(D) = 0.032$.

7. A student writes $\mathbb{P}(E \mid F) = \mathbb{P}(E \cap F)/\mathbb{P}(E)$. Diagnose. [L04]

The denominator must be the conditioning event F: $\mathbb{P}(E \mid F) = \mathbb{P}(E \cap F)/\mathbb{P}(F)$. Dividing by $\mathbb{P}(E)$ computes $\mathbb{P}(F \mid E)$ instead.

8. Given both children are not boys-and-boys (i.e. at least one girl), probability the *younger* is a girl? [L04]

Survivors $\{BG, GB, GG\}$; younger is a girl in BG, GG: $2/3$.

6. Mixed practice exam

Thirty-four multiple-choice questions in the style of the practice bank and the previous practice midterms: all four lectures shuffled, some "which correction is appropriate" audits, and a few interval-style answers. Work them closed-book with a calculator in about 75 minutes. The tag on each question says which lecture it comes from; the real exam will not tell you.

1. A locker code has four characters: two letters followed by two digits, with no letter repeated and no digit repeated. How many codes are possible? [L01]
(A) $26^2 \cdot 10^2 = 67600$ **(B)** $26 \cdot 25 \cdot 10 \cdot 9 = 58500$ **(C)** $\binom{26}{2}\binom{10}{2} = 14625$ **(D)** $26 \cdot 25 + 10 \cdot 9 = 740$ **(E)** $36 \cdot 35 \cdot 34 \cdot 33$

(B) 58500. Four ordered stages; letters deplete among letters, digits among digits. (A) allows repeats; (C) forgets order; (D) adds instead of multiplying; (E) lets letters and digits mix positions.

2. Five distinct books are arranged on a shelf. How many arrangements have two particular books next to each other? [L01]
(A) 24 (B) 48 (C) 60 (D) 96 (E) 120

(B) 48. Glue the pair into one block: 4 objects, $4! = 24$ orders, times 2 for the order within the block.

3. What is the coefficient of a^2b^4 in $(a-2b)^6$? [L01 · HW1]
(A) 15 (B) -240 (C) 240 (D) -60 (E) 96

(C) 240. $\binom{6}{2} \cdot 1^2 \cdot (-2)^4 = 15 \cdot 16 = 240$. The even power kills the sign.

4. How many distinct arrangements of the letters of PEPPER? [L01]
(A) 720 (B) 120 (C) 60 (D) 30 (E) 6

(C) 60. Six letters, P×3, E×2, R×1: $6!/(3!2!1!) = 720/12 = 60$.

5. Twelve distinct workers are assigned to three labelled shifts (morning, afternoon, night) of four each. How many assignments? [L01]
(A) $\binom{12}{4} = 495$ **(B)** 3^{12} **(C)** $\dfrac{12!}{(4!)^3} = 34650$ **(D)** $\dfrac{12!}{4!}$ **(E)** $12 \cdot 11 \cdot 10 \cdot 9$

(C) 34650. Multinomial $\binom{12}{4,4,4}$, interpretation 1 (labelled groups). (B) would let shift sizes vary.

6. A four-person committee is chosen from 6 men and 5 women. How many committees contain at least one woman? [L01]
(A) $\binom{11}{4} = 330$ **(B)** $\binom{5}{1}\binom{10}{3} = 600$ **(C)** $\binom{11}{4} - \binom{6}{4} = 315$ **(D)** $\binom{5}{4} = 5$ **(E)** $\binom{5}{1}\binom{6}{3} = 100$

(C) 315. Complement of "no women." (B) overcounts: a committee with two women is counted twice. (E) is "exactly one woman."

7. A committee of three is chosen uniformly from 4 statistics majors and 3 operations-research majors. Probability all three share a major? [L01 · L03]
(A) $1/35$ **(B)** $4/35$ **(C)** $1/7$ **(D)** $2/7$ **(E)** $6/35$

(C) 1/7. $\left[\binom{4}{3} + \binom{3}{3}\right]/\binom{7}{3} = (4+1)/35 = 1/7$.

8. An inbox records the number of emails received between noon and 1 p.m. Which describes a natural sample space? [L02]
(A) $\{0, 1\}$ **(B)** $\{0, 1, 2, \ldots\}$**, countably infinite (C)** $[0, 60]$**, continuous (D)** $\{1, 2, \ldots, 60\}$**, finite (E)** $\{\text{email}, \text{no email}\}$

(B). A count with no natural upper bound. A lifetime or a duration would be continuous.

9. Two dice are rolled and only the sum is recorded, $S = \{2, \ldots, 12\}$. Which event *cannot* be represented as a subset of this S? [L02]
(A) The sum is even (B) The sum exceeds 9 (C) The dice show doubles (D) The sum is 7 (E) The sum is at most 4

(C). Sum 8 arises from $(4, 4)$ and from $(2, 6)$; the record cannot tell them apart.

10. Which expression means "at most one of A and B occurs"? [L02]
(A) $A \cup B$ **(B)** $A^c \cap B^c$ **(C)** $(A \cap B)^c$ **(D)** $(A \cap B^c) \cup (A^c \cap B)$ **(E)** $A \cap B$

(C). At most one = not both. (D) is exactly one (excludes "neither"); (B) is neither.

11. Which is equal to $[(A \cap B) \cup C]^c$? [L02]
(A) $(A^c \cup B^c) \cap C^c$ **(B)** $(A^c \cap B^c) \cup C^c$ **(C)** $A^c \cap B^c \cap C^c$ **(D)** $(A^c \cup B^c) \cup C^c$ **(E)** $(A \cap B)^c \cup C$

(A). Outer union $\to$ intersection of complements: $(A \cap B)^c \cap C^c$; then inner intersection $\to$ $A^c \cup B^c$.

12. $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{1, 2, 3\}$, $F = \{4, 5\}$, $G = \{4, 5, 6, 7, 8\}$. Which is correct? [L02 · HW3]
(A) E and F are complements (B) E and G are disjoint but not complements (C) E and G are complements (D) F and G are disjoint (E) E and F are not disjoint

(C). $E \cap G = \emptyset$ and $E \cup G = S$. E, F are disjoint but $E \cup F \neq S$.

13. Simplify $(A \cup B) \cap (A \cup B^c)$. [L02]
(A) A **(B)** B **(C)** $A \cap B$ **(D)** $A \cup B$ **(E)** S

(A). Distributive: $A \cup (B \cap B^c) = A \cup \emptyset = A$.

14. Four outcomes are assigned masses 0.2, 0.3, 0.4, 0.2. Which audit is correct? [L03]
(A) Valid: each lies in $[0, 1]$ (B) Invalid: the masses sum to 1.1, contradicting normalisation and finite additivity (C) Invalid: outcomes must be equally likely (D) Valid: additivity only applies to infinite sequences (E) Cannot decide without data

(B). Coherence is a joint condition. (C), (D), (E) are all false statements about the axioms.

15. $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.4$, $\mathbb{P}(A \cup B) = 0.7$. Which ordered pair gives $\mathbb{P}(A \cap B)$ and $\mathbb{P}(\text{exactly one of } A, B)$? [L03]
(A) (0.2, 0.5) **(B)** (0.2, 0.7) **(C)** (0.3, 0.4) **(D)** (0.1, 0.6) **(E)** (0.2, 0.3)

(A). Overlap $0.5 + 0.4 - 0.7 = 0.2$; exactly one $= 0.7 - 0.2 = 0.5$.

16. $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B) = 0.6$. **Which value *could* be** $\mathbb{P}(A \cup B)$? [L03]
(A) 0.25 (B) 0.55 (C) 0.75 (D) 0.95 (E) 0.18

(C). Feasible interval $[\max(0.3, 0.6),\ \min(1, 0.9)] = [0.6, 0.9]$. Only 0.75 lies in it.

17. $\mathbb{P}(A) = 0.5, \mathbb{P}(B) = 0.4, \mathbb{P}(C) = 0.3$, $\mathbb{P}(A \cap B) = 0.2, \mathbb{P}(A \cap C) = 0.15, \mathbb{P}(B \cap C) = 0.1$, $\mathbb{P}(A \cap B \cap C) = 0.05$. **Probability none occurs?** [L03]
(A) 0.15 (B) 0.20 (C) 0.25 (D) 0.30 (E) 0.80

(B). Union $= 1.2 - 0.45 + 0.05 = 0.80$; none $= 0.20$.

18. Among 150 sessions, $|A| = 60, |B| = 50, |C| = 40$, $|A \cap B| = 20, |A \cap C| = 15, |B \cap C| = 10$, $|A \cap B \cap C| = 5$. **Probability a uniformly chosen session is in exactly one of the three?** [L03 · HW4]
(A) 0.30 (B) 0.40 (C) 0.50 (D) 0.60 (E) 0.73

(C). A only $= 60 - 20 - 15 + 5 = 30$; B only $= 50 - 20 - 10 + 5 = 25$; C only $= 40 - 15 - 10 + 5 = 20$. $75/150 = 0.5$.

19. A wheel has sectors of $45°, 45°, 90°, 180°$**; the first three are red. The pointer is equally likely to stop at any angle. A student says** $\mathbb{P}(\text{red}) = 3/4$. **Which is correct?** [L03]
(A) $3/4$, **three of four sectors are red (B)** $1/2$; **equal likelihood applies to angle, not sectors (C)** $1/4$ **(D)** $1/3$ **(E) Cannot be determined**

(B). Red covers $180/360$.

20. Ten raffle tickets, three are drawn uniformly without replacement. You hold one ticket. $\mathbb{P}(\text{you win})$? [L03]
(A) $1/10$ **(B)** $1/3$ **(C)** $\binom{9}{2}/\binom{10}{3} = 3/10$ **(D)** $1 - (9/10)^3$ **(E)** $3/\binom{10}{3}$

(C). Same as the tablet giveaway: k/n. (D) treats draws as with replacement.

21. Thirty people, birthdays uniform over 365 days. Which expression is $\mathbb{P}(\text{at least two share a birthday})$? [L03]
(A) $\dfrac{365 \cdot 364 \cdots 336}{365^{30}}$ **(B)** $1 - \dfrac{365 \cdot 364 \cdots 336}{365^{30}}$ **(C)** $\dfrac{\binom{30}{2}}{365}$ **(D)** $1 - \dfrac{\binom{365}{30}}{365^{30}}$ **(E)** $\dfrac{30}{365}$

(B). (A) is "all different." (D) mixes an unordered numerator with an ordered denominator.

22. Five cards from a standard deck. Probability all five are the same suit? [L03]
(A) $\dfrac{\binom{13}{5}}{\binom{52}{5}}$ **(B)** $\dfrac{4\binom{13}{5}}{\binom{52}{5}}$ **(C)** $\dfrac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{\binom{52}{5}}$ **(D)** $\left(\frac{1}{4}\right)^5$ **(E)** $\dfrac{4}{\binom{52}{5}}$

(B) ≈ 0.00198. Choose the suit (4), then 5 of its 13 cards. (C) mixes ordered numerator with unordered denominator.

23. 200 students: STAN majors 50 attended the review, 30 did not; non-STAN 40 attended, 80 did not. Which ordered pair gives $\mathbb{P}(\text{STAN} \mid \text{attended})$ **and** $\mathbb{P}(\text{attended} \mid \text{STAN})$? [L04]
(A) $(50/80,\ 50/90)$ **(B)** $(50/90,\ 50/80)$ **(C)** $(50/200,\ 50/200)$ **(D)** $(90/200,\ 80/200)$ **(E)** $(50/90,\ 90/200)$

(B) $= (0.556,\ 0.625)$. Attendees total 90; STAN majors total 80.

24. $\mathbb{P}(A) = 0.6$, $\mathbb{P}(B \mid A) = 0.5$, $\mathbb{P}(B) = 0.45$. $\mathbb{P}(A \mid B)$? [L04]
(A) 0.30 (B) 0.50 (C) 0.6667 (D) 0.75 (E) 0.27

(C). $\mathbb{P}(A \cap B) = 0.30$; $0.30/0.45 = 2/3$.

25. Two children, ordered, four outcomes equally likely. Which ordered pair gives $\mathbb{P}(\text{both girls} \mid \text{older is a girl})$ and $\mathbb{P}(\text{both girls} \mid \text{at least one girl})$? [L04]
(A) $(1/2, 1/2)$ **(B)** $(1/3, 1/3)$ **(C)** $(1/2, 1/3)$ **(D)** $(1/3, 1/2)$ **(E)** $(1/4, 1/3)$

(C). Survivors $\{GB, GG\}$ versus $\{BG, GB, GG\}$.

26. An urn has 4 red and 6 blue balls. Three are drawn in order without replacement. $\mathbb{P}(\text{red, blue, blue})$? [L04]
(A) $\frac{4}{10} \cdot \frac{6}{10} \cdot \frac{6}{10}$ **(B)** $\frac{4}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} = \frac{1}{6}$ **(C)** $\frac{4}{10} \cdot \frac{6}{9} \cdot \frac{6}{8}$ **(D)** $\binom{4}{1}\binom{6}{2}/\binom{10}{3}$ **(E)** $\frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8}$

(B). Chain rule with depletion. (D) is the unordered "one red, two blue" event, three times larger.

27. A student writes $\mathbb{P}(R_1 \cap R_2) = \frac{4}{10} \cdot \frac{4}{10}$ for two draws without replacement from 4 red and 6 blue. Which correction is appropriate? [L04]
(A) Keep it; draws are independent (B) $\frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15}$ **(C)** $\frac{4}{10} + \frac{3}{9}$ **(D)** $\frac{4}{10} \cdot \frac{3}{10}$ **(E)** $\binom{4}{2}/\binom{10}{2}$ **is the only valid form**

(B). The second factor is $\mathbb{P}(R_2 \mid R_1)$ with 3 reds among 9. (E) gives the same number but is not the "only" valid form.

28. Two sections chosen with probability $1/2$ each. Section 01 is 40% actuarial and an actuarial student replies with probability 0.5; Section 02 is 60% actuarial with reply probability 0.25. Non-actuarial students never reply. $\mathbb{P}(\text{reply})$? [L04]
(A) 0.100 (B) 0.075 (C) 0.175 (D) 0.350 (E) 0.250

(C). $0.5 \cdot 0.4 \cdot 0.5 + 0.5 \cdot 0.6 \cdot 0.25 = 0.10 + 0.075$.

29. Two fair dice. Given at least one shows a 6, probability the sum is 8? [L04]
(A) $2/36$ **(B)** $2/11$ **(C)** $1/6$ **(D)** $5/36$ **(E)** $2/12$

(B). Eleven outcomes contain a 6; $(6, 2)$ and $(2, 6)$ sum to 8.

30. A committee of three from 5 statistics and 2 OR majors. Given at least one OR major is on it, probability exactly one is? [L04]
(A) $20/35$ **(B)** $4/5$ **(C)** $1/5$ **(D)** $25/35$ **(E)** $2/7$

(B). At least one OR: $\binom{7}{3} - \binom{5}{3} = 25$. Exactly one: $\binom{2}{1}\binom{5}{2} = 20$. $20/25$.

31. $\mathbb{P}(A \mid B) = 0.6$, $\mathbb{P}(B) = 0.5$, $\mathbb{P}(A) = 0.45$. **Which ordered pair gives $\mathbb{P}(A^c \mid B)$ and $\mathbb{P}(B \mid A)$?** [L04]
(A) $(0.4, 0.6)$ **(B)** $(0.4, 2/3)$ **(C)** $(0.55, 0.3)$ **(D)** $(0.4, 0.3)$ **(E)** $(0.6, 2/3)$

(B). Complement inside B: 0.4. $\mathbb{P}(A \cap B) = 0.6 \cdot 0.5 = 0.30$; $\mathbb{P}(B \mid A) = 0.30/0.45 = 2/3$.

32. Nine toppings including X and Y. A pizza has exactly four toppings and never both X and Y. The number N of allowed pizzas lies in: [L01]

(A) $(0, 35]$ **(B)** $(35, 70]$ **(C)** $(70, 100]$ **(D)** $(100, \infty)$ **(E) none of these**

(D). Neither: $\binom{7}{4} = 35$. Exactly one: $2\binom{7}{3} = 70$. $N = 105$.

33. John attends with probability 0.7. Given John attends, Kate attends with probability 0.6; given John does not, 0.5. Probability exactly one attends lies in: [L04]

(A) $(0, 0.3]$ **(B)** $(0.3, 0.5]$ **(C)** $(0.5, 0.7]$ **(D)** $(0.7, 1]$ **(E) It cannot be found without independence**

(B). $0.7 \cdot 0.4 + 0.3 \cdot 0.5 = 0.28 + 0.15 = 0.43$. No independence is needed; the tree supplies the conditionals.

34. A student claims that for any events $E \subseteq F$, $\mathbb{P}(F \mid E) = \mathbb{P}(E \mid F)$. Which audit is correct?

[L04 · L03]

(A) True, both equal $\mathbb{P}(E \cap F)$ **(B)** $\mathbb{P}(F \mid E) = 1$ **always, but** $\mathbb{P}(E \mid F) = \mathbb{P}(E)/\mathbb{P}(F)$**, which is 1 only if** $\mathbb{P}(E) = \mathbb{P}(F)$ **(C) Both equal** $\mathbb{P}(E)$ **(D)** $\mathbb{P}(E \mid F) = 1$ **always (E) Neither can be computed**

(B). $E \subseteq F$ means $E \cap F = E$, so $\mathbb{P}(F \mid E) = \mathbb{P}(E)/\mathbb{P}(E) = 1$ and $\mathbb{P}(E \mid F) = \mathbb{P}(E)/\mathbb{P}(F) \leq 1$. Example: both hearts given at least one heart $= 2/15$, not 1.

7. One-screen cram sheet

There is no formula sheet in the exam. Everything below must be in your head.

COUNTING. Product rule $n_1 n_2 \cdots n_r$. Ordered, no repeats: $P(n,r) = \dfrac{n!}{(n-r)!}$. Unordered: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{0} = 1$. Subsets: 2^n. Repeats allowed: n^r.

MULTINOMIAL. $\binom{n}{n_1,\ldots,n_r} = \dfrac{n!}{n_1!\cdots n_r!}$. Labelled groups of given sizes, *or* arrangements with n_i alike. BANANAS $= 7!/(3!2!) = 420$.

BINOMIAL THEOREM. $(x+y)^n = \sum \binom{n}{r} x^r y^{n-r}$. Coefficient of $x^r y^{n-r}$ in $(ax+by)^n$ is $\binom{n}{r} a^r b^{n-r}$. Keep the scalars and the sign.

CONSTRAINTS. Forbidden pair together: total minus (both in). Feud among 5 from 8: $\binom{6}{5} + 2\binom{6}{4} = 36$. Adjacent items: glue into a block. "At least one": complement. Disjoint patterns: add.

SAMPLE SPACE. Record enough detail for every event you need. Sums and counts lose *which* component. Finite / countably infinite $\{1,2,\ldots\}$ / continuous $[0,\infty)$. Event = subset; $\emptyset$ impossible, S certain.

EVENT ALGEBRA. $\cup$ at least one; $\cap$ all; $E^c = S \setminus E$; $E \setminus F = E \cap F^c$. Disjoint: $E \cap F = \emptyset$. Complements: disjoint *and* $E \cup F = S$. $E \cap F \subseteq E$.

DE MORGAN. $(E \cup F)^c = E^c \cap F^c$; $(E \cap F)^c = E^c \cup F^c$. Nested: apply outside first. Distributive: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$.

PHRASES. Exactly one: $(E \cap F^c) \cup (E^c \cap F)$. Exactly two of three: three terms each with one complement. At least two: $(E \cap F) \cup (E \cap G) \cup (F \cap G)$. None: $E^c \cap F^c \cap G^c$.

AXIOMS. $\mathbb{P}(E) \geq 0$; $\mathbb{P}(S) = 1$; disjoint $\Rightarrow \mathbb{P}(\bigcup E_i) = \sum \mathbb{P}(E_i)$. Audit: nonnegativity first, then do the masses sum to 1? Equal masses are an *assumption*, never required.

CONSEQUENCES. $\mathbb{P}(\emptyset) = 0$; $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$; $E \subseteq F \Rightarrow \mathbb{P}(E) \leq \mathbb{P}(F)$; $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$; $\mathbb{P}(E) = |E|/|S|$ only when equally likely.

INCLUSION–EXCLUSION. $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$. Three: singles − pairs + triple. Exactly one of two $= \mathbb{P}(E \cup F) - \mathbb{P}(E \cap F)$. A only $= |A| - |AB| - |AC| + |ABC|$. AB only $= |AB| - |ABC|$.

BOUNDS. $\mathbb{P}(E \cap F) \leq \min$; $\max \leq \mathbb{P}(E \cup F) \leq \min\{1, \mathbb{P}(E) + \mathbb{P}(F)\}$; also $\mathbb{P}(E \cap F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1$.

BIRTHDAY. $\mathbb{P}(\text{all distinct}) = \dfrac{365 \cdot 364 \cdots (365-n+1)}{365^n}$; shared $= 1-$that. $n = 23$ first exceeds $1/2$. Same shape for m labels and n items.

CARDS. $\binom{52}{5} = 2{,}598{,}960$. Full house $13\binom{4}{3} \cdot 12\binom{4}{2} = 3744$. Two pairs $\binom{13}{2}\binom{4}{2}^2 \cdot 44 = 123552$. Ordered role ranks $13 \cdot 12$; same-role ranks $\binom{13}{2}$.

SAMPLING. k of n chosen: $\mathbb{P}(\text{you}) = k/n = \binom{n-1}{k-1}/\binom{n}{k}$. Without replacement: $\binom{a}{i}\binom{b}{j}/\binom{a+b}{i+j}$ or chain rule. With replacement: independent factors, repeats possible.

Conditional. $\mathbb{P}(E \mid F) = \dfrac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$, $\mathbb{P}(F) > 0$. Restrict, then renormalise. Table: divide by the row or column named after the bar. $\mathbb{P}(E \mid F) \neq \mathbb{P}(F \mid E)$.

Multiplication and chain. $\mathbb{P}(E \cap F) = \mathbb{P}(F)\mathbb{P}(E \mid F) = \mathbb{P}(E)\mathbb{P}(F \mid E)$. $\mathbb{P}(E_1 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2 \mid E_1)\cdots$. Without replacement the later factors change.

Reverse the bar. Given $\mathbb{P}(E), \mathbb{P}(B \mid E), \mathbb{P}(B)$: first $\mathbb{P}(E \cap B) = \mathbb{P}(E)\mathbb{P}(B \mid E)$, then $\mathbb{P}(E \mid B) = \mathbb{P}(E \cap B)/\mathbb{P}(B)$. Complement inside a universe: $\mathbb{P}(E^c \mid B) = 1 - \mathbb{P}(E \mid B)$.

Trees. Edges are conditionals; branches from one node sum to 1. Multiply along a path; add across disjoint paths. "Exactly one" = two paths added.

Traps that cost points. Scalars in binomial coefficients (-448, not 56). Ordered vs unordered mixing. $|A \cap B|$ miscounted in dice unions (list them). Sum-only sample spaces. "At least two" vs "exactly two." Wrong denominator after the bar. Ratio without equal likelihood.

Last check. Is the answer in $[0, 1]$? Does the count match the structure (order? repeats? labelled?)? Does a with/without replacement pair move in the right direction? Bubble the sheet.

Built September 15, 2026 from the Lecture 1–4 decks, Homework 1–4, the cumulative practice bank, and the Midterm 1 information sheet of STOR 435.002 (Fall 2026, UNC). Personal study material.